

A tutorial on resolvent methods

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What is resolvent analysis?

Why would I do this?

How do I do this?

Motivation

Unlike DMD or POD,
we seek **equations-driven** (not data-driven) results.

Some key works

- ▶ Farrel, Trefethen, Schmid & Henningson etc 1980's-2000's
(pseudospectra, TG)
- ▶ Bamieh & Jovanovic JFM 2005
(LNS transfer function response, laminar)
- ▶ ONERA, Hwang & Cossu, Gallaire ...
(optimal response)
- ▶ McKeon & Sharma JFM 2010
(turbulent mean, resolvent modes, critical layer, NL)
- ▶ Mezic ND 2005 / Schmid JFM 2010 / Rowley e.a. JFM 2009
(Koopman modes / DMD)

Claims

- ▶ Resolvent modes make predictions about dominant structures at freqs
- ▶ Directly related to equations of motion
- ▶ They can be used to predict DMD modes
- ▶ They can be used for flow estimation from point data

An expansion around the turbulent attractor I

Let $\mathbf{u}(t)$ be the state, and the Navier-Stokes equations be written

$$\dot{\mathbf{u}}(t) = \mathbf{f}(\mathbf{u}(t)).$$

In the “long run”, decompose the state as

$$\mathbf{u}(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{i\omega t} \hat{\mathbf{u}}(\omega) d\omega.$$

Notice that the equation corresponding to $\omega = 0$ is the mean equation

$$0 = \int_{-\infty}^{\infty} \mathbf{f}(\mathbf{u}) dt$$

with $\hat{\mathbf{u}}(0)$ the mean.

An expansion around the turbulent attractor II

The expansion of the dynamics about this mean (and subtracting this mean equation) is

$$\begin{aligned}\dot{\mathbf{u}}(t) &= \left. \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right|_{\bar{\mathbf{u}}} \tilde{\mathbf{u}}(t) + \tilde{\mathbf{g}}(t) \\ &= \mathbf{L} \tilde{\mathbf{u}}(t) + \tilde{\mathbf{g}}(t)\end{aligned}$$

where $\tilde{\mathbf{g}}$ represents the second-order terms in the expansion of $\mathbf{f}$.

At a particular $\omega \neq 0$ we then have

$$\hat{\mathbf{u}}(\omega) = (\mathrm{i}\omega I - \mathbf{L})^{-1} \hat{\mathbf{g}}(\omega).$$

The second-order terms, rather than being truncated, act to excite the state.

McKeon & Sharma, JFM, 2010.

A low-rank basis

The resolvent $\mathbf{H}(\omega) = (\mathrm{i}\omega I - \mathbf{L})^{-1}$ is well-approximated by a projection $\Pi(\omega)$,

$$\hat{\mathbf{u}}(\omega) = \mathbf{H}(\omega)\hat{\mathbf{g}}(\omega) \simeq \Pi(\omega)\hat{\mathbf{g}}(\omega).$$

The SVD gives the optimal N -dimensional basis on which the velocity field evolves, in sense that $\|\mathbf{H}(\omega) - \Pi_N(\omega)\|_F$ is minimised.

The flow “lives” mostly in Π .

The ‘error’ is $\hat{\mathbf{u}}^\perp = \Pi_N^\perp \hat{\mathbf{g}}$.

McKeon & Sharma, JFM, 2010.

A basis for optimal projection; resolvent modes

$$\mathbf{H}\hat{\mathbf{g}} = \sum_{m=1}^{\infty} \psi_m(\mathbf{x}) \sigma_m \langle \phi_m(\mathbf{x}), \hat{\mathbf{g}}(\mathbf{x}) \rangle$$

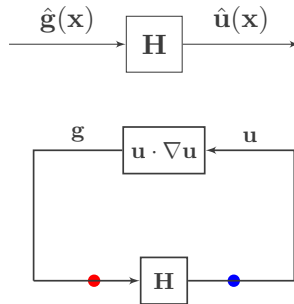
$$\langle \psi_m, \psi_{m'} \rangle = \delta_{m,m'}$$

$$\langle \phi_m, \phi_{m'} \rangle = \delta_{m,m'}$$

$$\sigma_1 \geq \sigma_2 \geq \dots$$

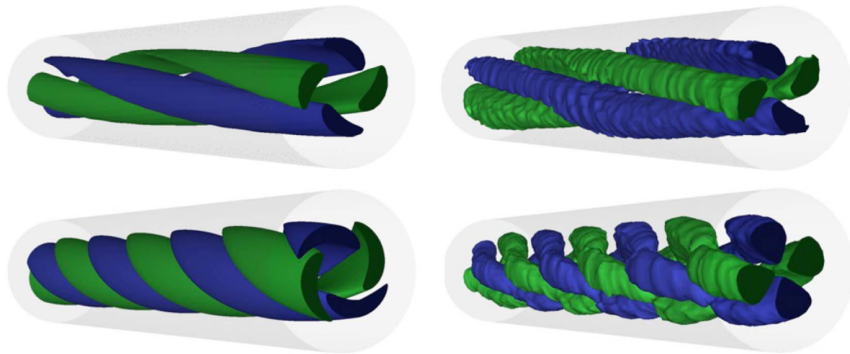
Each σ_m is a (real) gain, σ_1 is the maximum gain.

Velocity field response is $\psi_m(\mathbf{x})$.



McKeon & Sharma, JFM, 2010.

Why do DMD modes look like resolvent modes?



Gomez et al *PoF* 2014

“Paco’s pipe projection puzzle”

$n = 2$, $\omega/2\pi = 0.1826, 0.5497$, isosurfaces $1/3u_{max}$.

Relation between Koopman/DMD and resolvent modes

Comparing starting point of RM derivation,
we see that the resolvent relates the Koopman modes of $\mathbf{u}$ and $\mathbf{g}$

$$\hat{\mathbf{u}}(\omega; \mathbf{x}) = \mathbf{H}(\omega) \hat{\mathbf{g}}(\omega; \mathbf{x})$$

Sharma, Mezic, McKeon, PRF, 2016

DMD/Koopman: Schmid JFM 2010; Rowley et al JFM 2009

Interpretation

- ▶ Resolvent modes a good 'first guess' of DMD modes
- ▶ Allows ordered expansion of DMD modes,
projection of velocity DMD modes onto RM gives coefficients
- ▶ Expansion in RMs gives sensitivity of KMs to control / BC changes

LSQ estimation approach

Write velocity field at probe points as truncated sum over modes at desired frequencies,

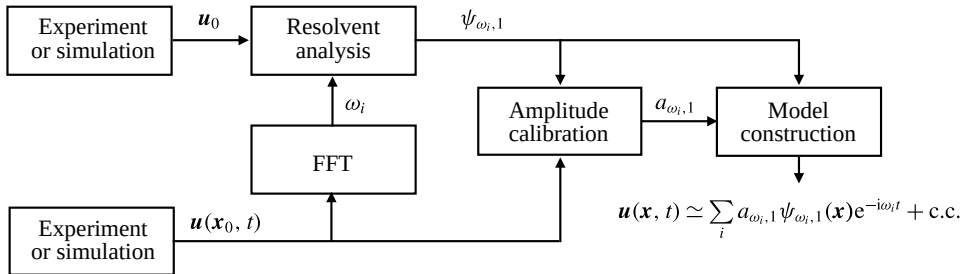
$$\mathbf{u}(\mathbf{x}_p, t) = \sum_{n=1}^N \sum_{m=1}^M \psi_{\omega_n, m}(\mathbf{x}_p) a_{\omega_n, m} e^{i\omega_n t} + c.c.$$

(taking just first response mode (if appropriate) simplifies the problem).

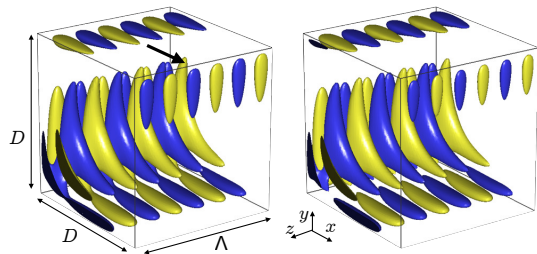
Solve for coefficients A in least-squares sense (pseudo-inverse),

$$\Psi(\mathbf{x}_p)A = U(\mathbf{x}_p).$$

Gomez et al, JFM 2016, Beneddine et al, JFM 2016



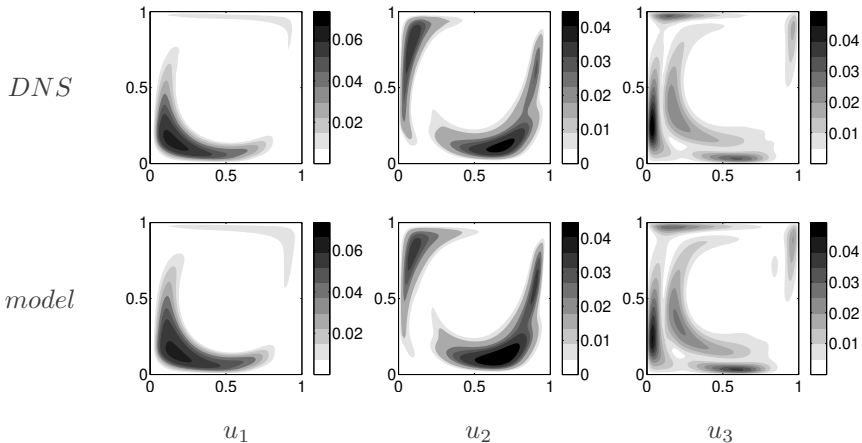
Example: resolvent modes in a lid-driven cavity



- ▶ $Re = 1200, \Lambda = 0.945D$
- ▶ Nonlinear, low-dimensional behaviour
- ▶ Three dominant wavenumbers:
 $\beta = 0, 3, 6$
- ▶ Three dominant frequencies:
 $\omega = 0, 0.76, 1.52$
- ▶ spectral-*hp* 2D $\times$ Fourier (semtex)

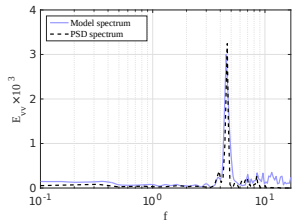
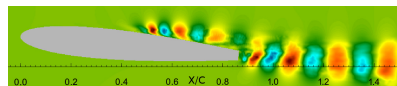
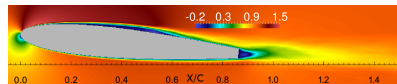
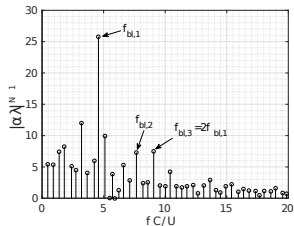
Gomez et al, JFM v798 2016

Reconstruction, $\mathbf{x}_p = (0.1, 0.1, 0)$; (RMS fluctuations)



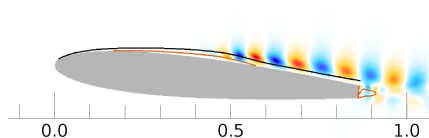
Gomez et al, JFM v798 2016

DMD vs resolvent mode; Flow estimation, for BTE aerofoil



$X/C = 0.47, Y/C = 0.035$.

(probe at $X/C = 0.62, Y/C = 0.022$).



Thomareis (Papadakis), Imperial thesis, 2017; Thomareis & Papadakis, PoF 2017

Numerics: Matrix-forming (1D, 2D)

1D problems (two invariant spatial directions) are small; modes found from full SVD.

$$[U, S, V] = \text{svd}(H)$$

For 2D, use that the SVD is equivalent to an eigenvalue problem,

$$H^*(\omega)QH(\omega)\phi_m(\omega) = \sigma^2(\omega)\phi_m(\omega)$$

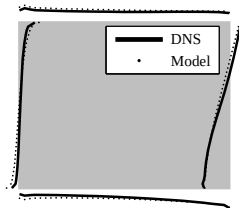
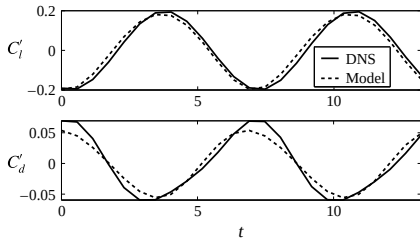
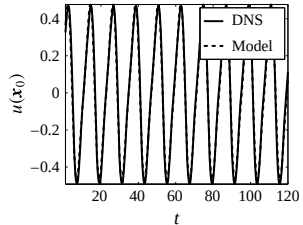
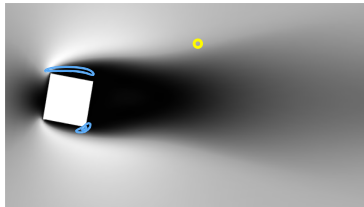
The matrix-forming approach can then use Arnoldi/Lanczos etc, via ARPACK.

Numerics: Matrix-free Timestepping

- 1: Allow complex velocity fields
- 2: loop until converged
- 3: March $\partial_t \mathbf{u} = L\mathbf{u} + \hat{\mathbf{f}}e^{i\omega t}$, $\mathbf{u}(0) = \mathbf{u}_0$ for n periods
- 4: $\sigma \leftarrow$ amplification
- 5: March $\partial_t \mathbf{v} = -L^*\mathbf{v} + \hat{\mathbf{u}}e^{i\omega t}$, $\mathbf{v}(0) = \mathbf{u}(nT)$
- 6: $\sigma \leftarrow$ amplification; $\hat{\mathbf{f}} = \mathbf{v}/\sigma^2$

Monokrousos et al, JFM 2010; Lu & Papadakis IJNMF, 2014; Gomez et al, JFM v804 2016

Timestepping example: square cylinder

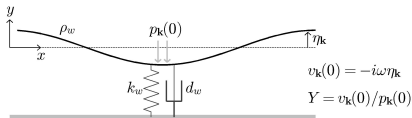


Gomez, Sharma, Blackburn, JFM v804 2016

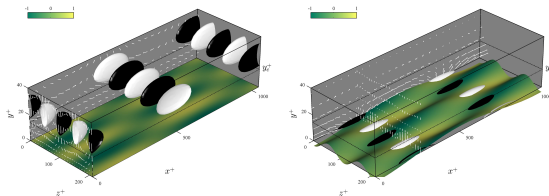
Compliant wall boundary conditions

Linearised BC represented as a complex admittance

$$Y = \frac{v_k}{p_k(0)}$$

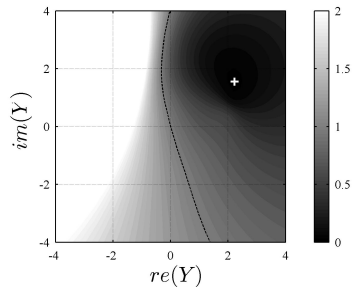


$$Re_\tau = 2000, k_x = 12, k_z = \pm 120, c^+ = 10$$



no control vs. optimised wall properties

Near-wall cycle requires negative damping



Integrated Re stress (drag) vs wall's complex admittance

Luhar, Sharma, McKeon, JFM 2015; JoT 2016

Conclusions

- ▶ DMD modes look like RM because RM look like Koopman modes
- ▶ Resolvent modes offer (linear-) optimal basis derived from equations
- ▶ Global coherence allows fixing of coefficients from point data
- ▶ Can use this to predict sensitivity of chaotic system to control

A tutorial on resolvent methods

Abstract

Arising from the interaction between control theory and more traditional fluid dynamics, resolvent analysis methods focus on the response of the fluid to dynamic disturbances. Many styles of this approach now exist, which variously consider that disturbance as a stochastic forcing, as an externally applied forcing, as internally generated Reynolds stresses, or as some combination thereof. In this tutorial paper, we shall introduce the theoretical viewpoint advanced by the authors in previous work. The approach is a complete and self-consistent restatement of the Navier-Stokes equations, which permits a series of progressively lower-dimensional approximations to be made. This viewpoint has a number of interesting characteristics that relate to other techniques. For example, the approach offers a more defensible alternative to eigenvalue stability calculations about a temporal-mean flow. The presentation will also highlight the natural links to the Dynamic Mode Decomposition and Koopman modes approaches. We shall illustrate the approach by reviewing some recent applications to passive and active flow control strategies, flow estimation, flow structure prediction, and flow spectra.