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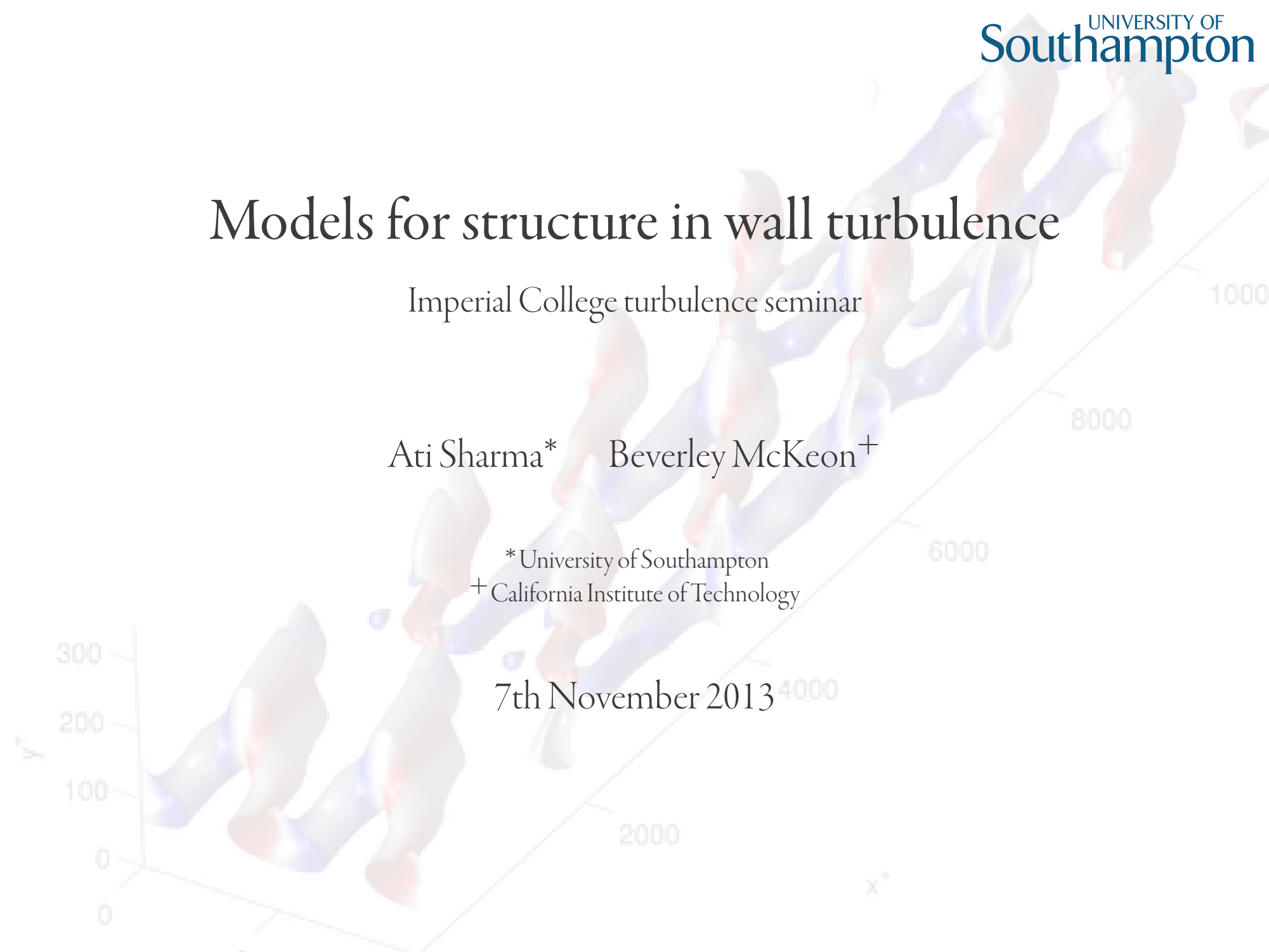
Models for structure in wall turbulence

Imperial College turbulence seminar

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*University of Southampton
+California Institute of Technology

7th November 2013



Overview

Introduction

Model formulation

Structure

Self-sustaining solutions

Conclusion

Source papers for this talk

- McKeon & Sharma JFM 2010 (model, VLSMs)
- **Sharma & McKeon JFM 2013** (about structure)
- McKeon & al PoF 2013 (experiment in BL, review / introduction)
- Moarref & al JFM 2013 (prediction of spectra, scaling of resolvent modes)

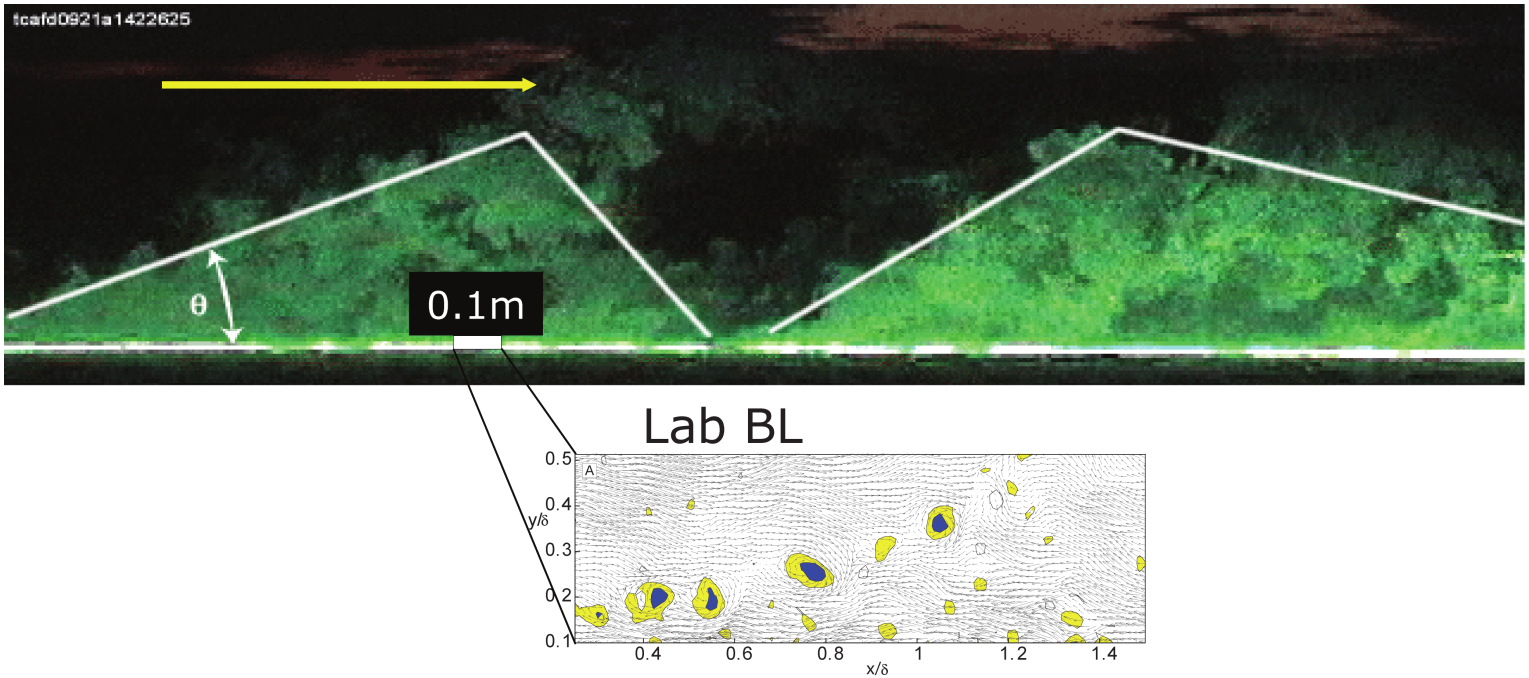
Coherent structure in wall turbulence

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...a three-dimensional region of the flow over which at least one fundamental flow variable [...] exhibits significant correlation with itself or with another variable over a range of space and/or time that is significantly larger than the smallest local scales of the flow

Robinson ARFM 1991

Coherent structure in wall turbulence



Adrian PoF 2007

Coherent structure in wall turbulence

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- Coherent structure means some patterns recur
- These patterns must be more important
- We want to use the equations, not simulation data
- What to throw away?

Are low order models feasible?

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- Navier-Stokes equations are infinite-dimensional
- but turbulence has features common across flows (structure)
- "Coherent structure" itself is a conceptual / qualitative / mental model
- So turbulence must admit more formal reduced order models that describe structure
- Keefe & al (JFM 1992) estimate $D_\lambda \simeq 780$ at $Re = 3200$

Some relevant work

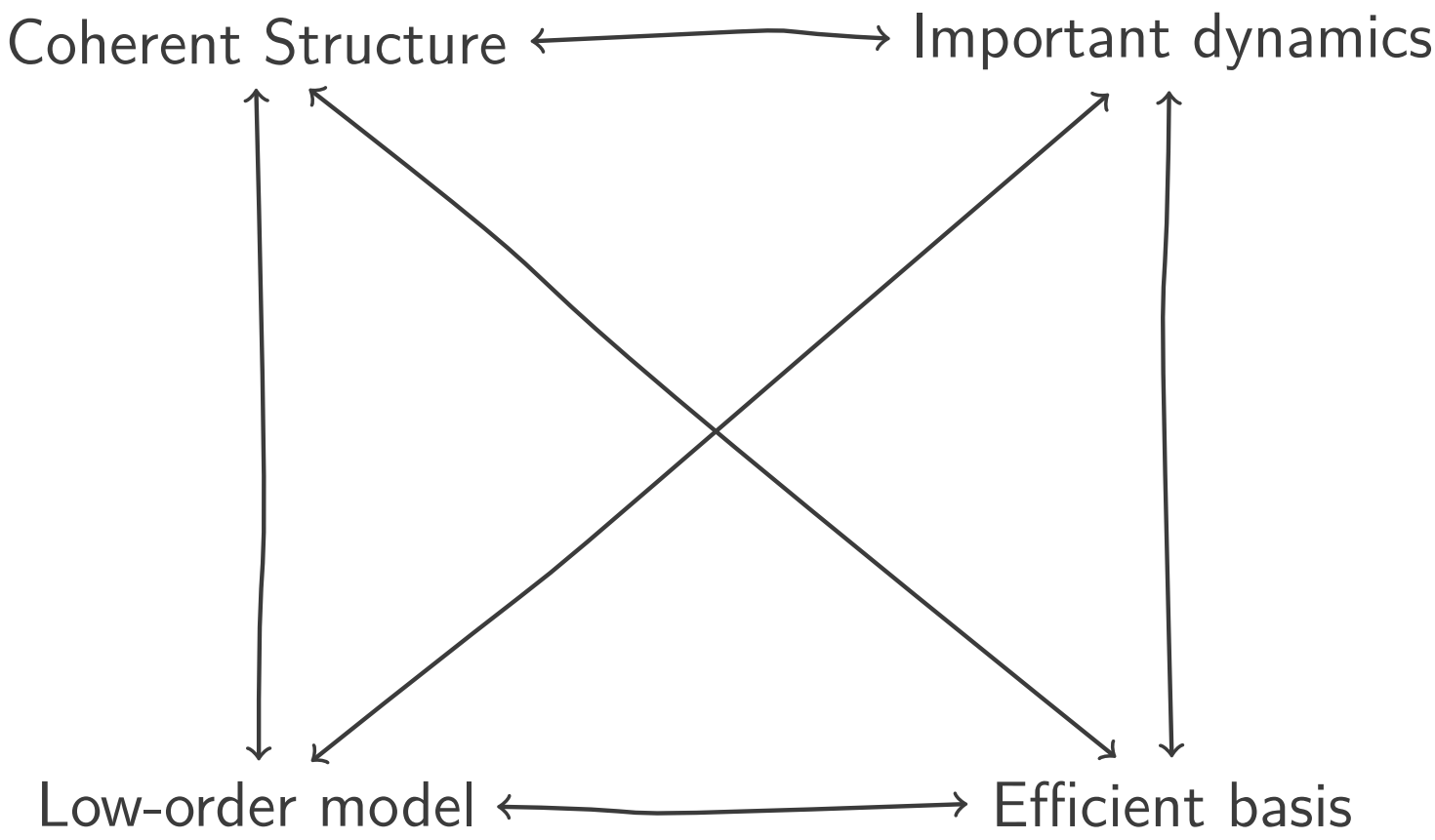
- POD (SVD of data)
- DMD (also from data)
- RDT
- Landahl's models
- time-domain linear methods (transient growth, GOP etc)
- modal (classical linear instability) theory
- asymptotic methods
- nonlinear exact solutions

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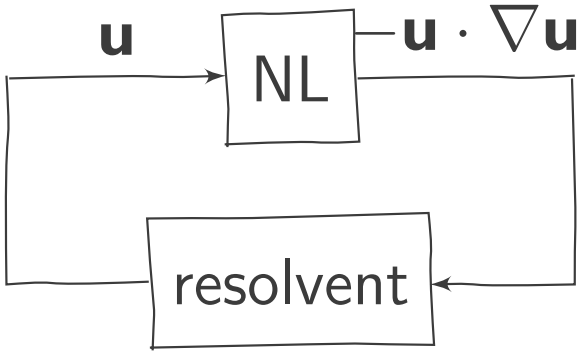
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Coherent Structure
Important dynamics
Low-order model
Efficient basis

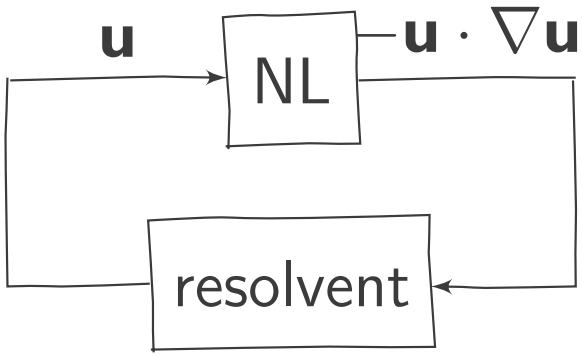
Structure is selective dynamics

- coherent structure is a manifestation of selective dynamics;



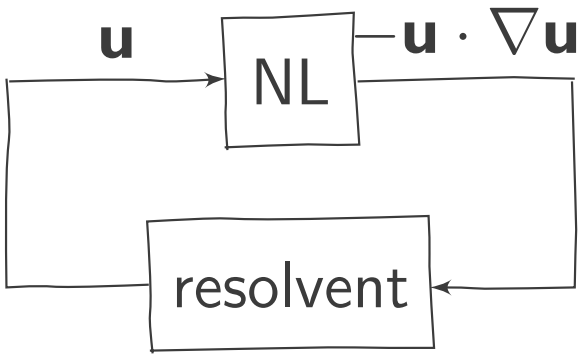
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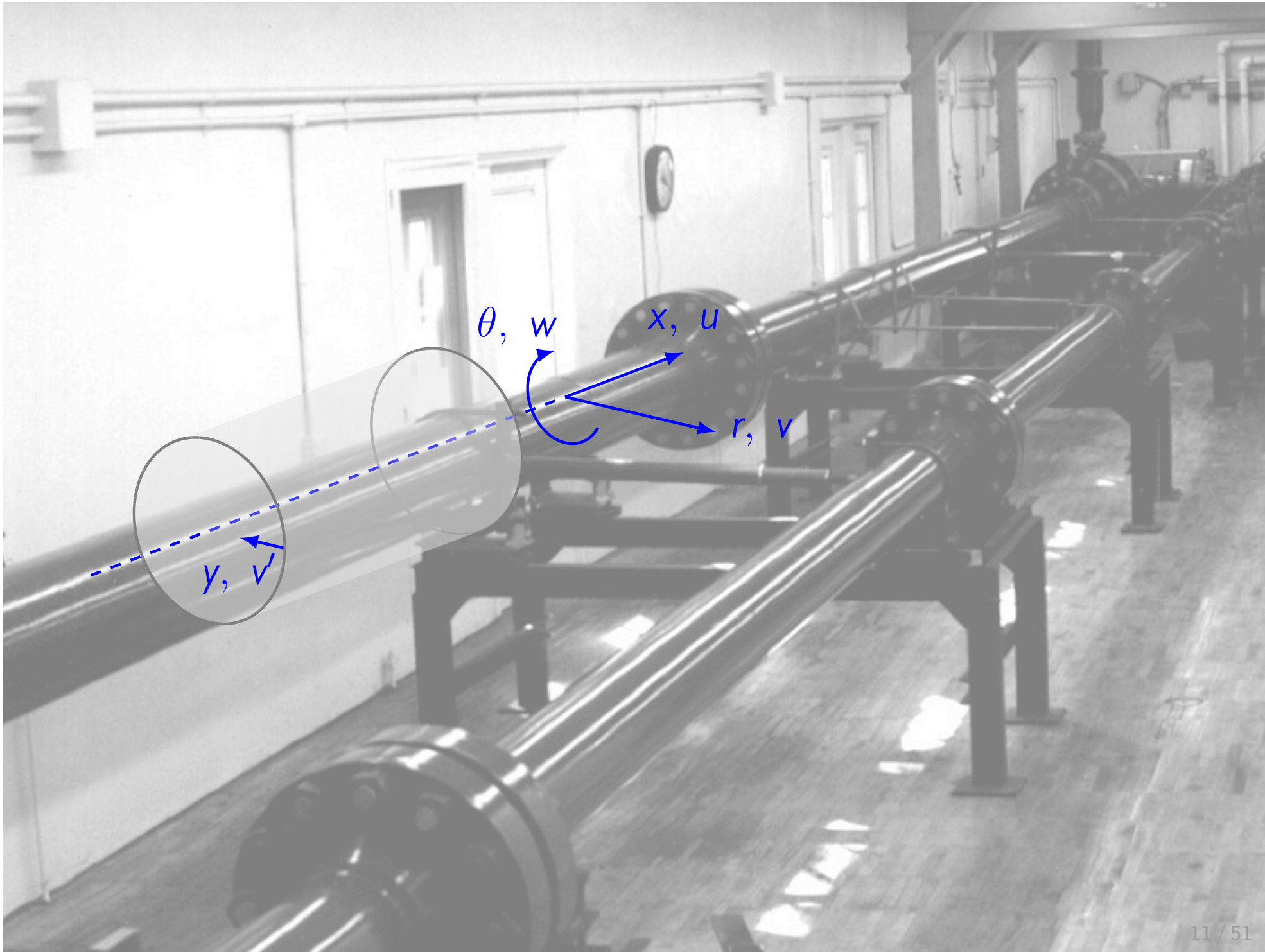
- coherent structure is a manifestation of selective dynamics;
- we can measure relative importance of dynamics by selectivity of the linear resolvent operator;

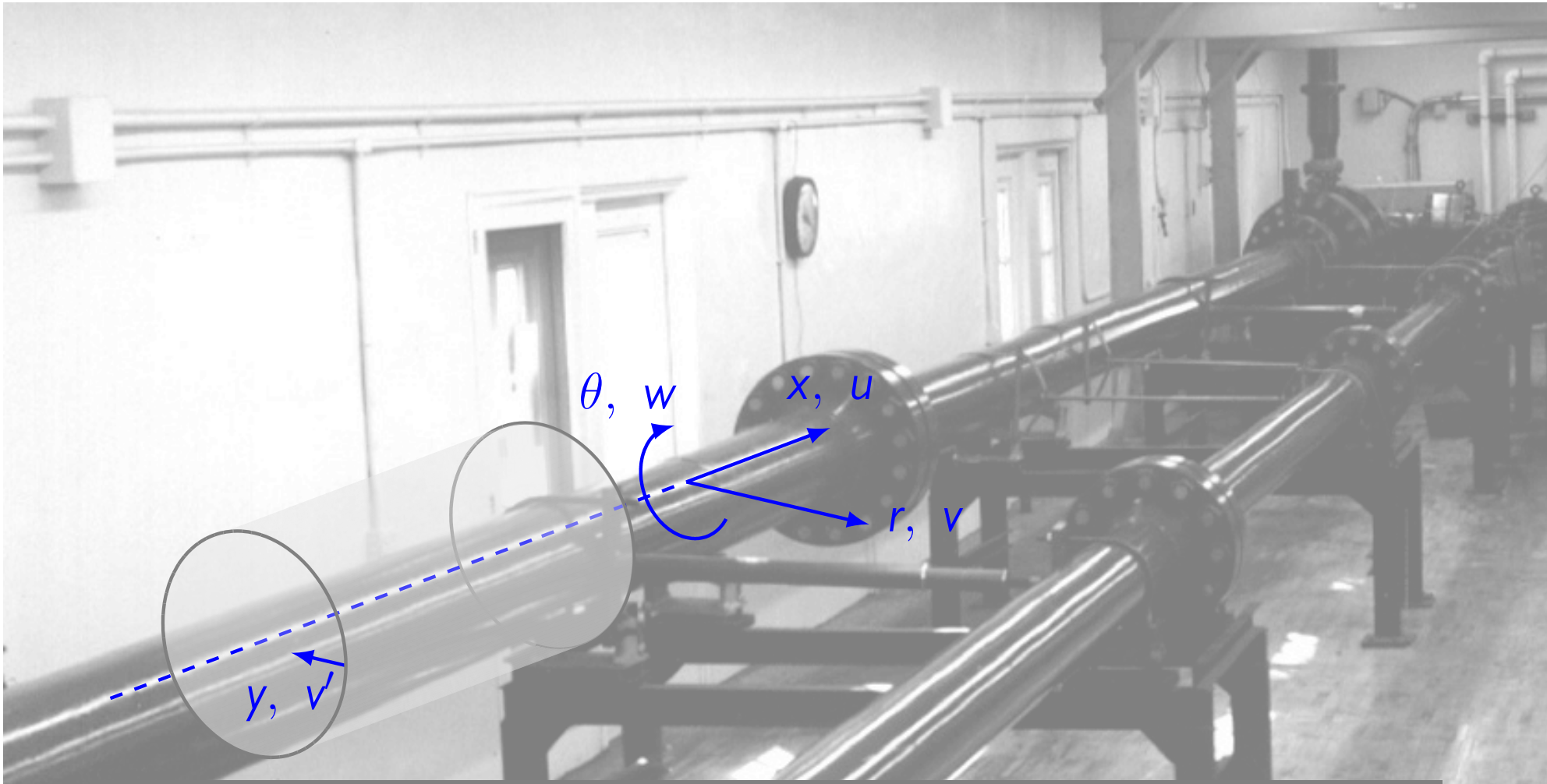


Structure is selective dynamics

- coherent structure is a manifestation of selective dynamics;
- we can measure relative importance of dynamics by selectivity of the linear resolvent operator;
- we can well describe this selection via singular value decomposition of resolvent in frequency-wavenumber domain




θ, w
x, u
r, v
y, v'



Fourier decomposition into radially-varying travelling waves

$$\mathbf{u}(r;x,\theta,t) = \sum_{k,n,\omega} \mathbf{u}(r;k,n,\omega)\mathrm{e}^{i(kx+\theta n-\omega t)}$$

NSE in Fourier domain

Fourier modes for velocity in all three homogeneous directions

$$\mathbf{u}(r; x, \theta, t) = \sum_{k,n,\omega} \mathbf{u_k}(r) \mathrm{e}^{i(kx+\theta n-\omega t)}$$

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subtract out steady-state / $\mathbf{u}_0$ / mean / $(k, n, \omega) = (0, 0, 0)$
and substitute for nonlinear term

$$\mathbf{f}(r; x, \theta, t) := -\left(\mathbf{u} - \mathbf{u}_0(r)\right) \cdot \nabla \left(\mathbf{u} - \mathbf{u}_0(r)\right)$$

$$\mathbf{f}(r; x, \theta, t) := \sum_{k,n,\omega} \mathbf{f_k}(r) \mathrm{e}^{i(kx+\theta n-\omega t)}$$

NSE in Fourier domain

- Just expressed field (and operators) as linear superposition of radially-varying travelling waves

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- Clear interpretation for linear operators formed about the mean profile – *this is **not** small perturbation analysis around a fixed point*

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- Clear interpretation for linear operators formed about the mean profile – *this is **not** small perturbation analysis around a fixed point*
- Provide closure problem by assuming $\mathbf{u}_0$ (or any other $\mathbf{u}_\mathbf{k}$)
- No need for eddy viscosity because $\mathbf{f}_0$ known

Steady-state equation

The equation for $(k, n, \omega) = 0$ gives the mean

$$0 = \mathbf{f}_0(r) - \mathbf{u}_0(r) \cdot \nabla \mathbf{u}_0(r) + Re^{-1} \nabla^2 \mathbf{u}_0(r)$$

Steady-state equation

Re stress gradients supporting mean

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time-space ave velocity

Equation for fluctuations at (k, n, ω)

$$
\begin{aligned}
-i\omega \mathbf{u}_{\mathbf{k}}(r) = & -\mathbf{u}_0(r) \cdot \nabla \mathbf{u}_{\mathbf{k}}(r) \\
& -\mathbf{u}_{\mathbf{k}}(r) \cdot \nabla \mathbf{u}_0(r) \\
& +Re^{-1} \nabla^2 \mathbf{u}_{\mathbf{k}}(r) - \nabla p_{\mathbf{k}}(r) \\
& +\mathbf{f}_{\mathbf{k}}(r)
\end{aligned}
$$

Equation for fluctuations at (k, n, ω)

$$-i\omega \mathbf{u}_{\mathbf{k}}(r) = \mathrm{L}_{\mathbf{k}}(r)\mathbf{u}_{\mathbf{k}}(r) + \mathbf{f}_{\mathbf{k}}(r)$$

Equation for fluctuations at (k, n, ω)

$$\mathbf{u_k}(r) = (-i\omega - \mathrm{L_k}(r; \mathbf{u}_0))^{-1} \mathbf{u_k}(r) \mathbf{f_k}(r)$$

Equation for fluctuations at (k, n, ω)

interaction between scales

$$\mathbf{u_k}(r) = \left(-i\omega - \mathrm{L_k}(r; \mathbf{u}_0)\right)^{-1} \mathbf{f_k}(r)$$

linear operator

Equation for fluctuations at (k, n, ω)

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linear operator

$\mathbf{u_k}$ requires $\mathbf{u}_0$ requires $\mathbf{u_k}$... close by assuming $\mathbf{u}_0$

Equation for fluctuations at (k, n, ω)

$$\mathbf{u_k}(r) = \underbrace{\left(-i\omega - \mathrm{L_k}(r; \mathbf{u}_0)\right)^{-1}}_{\textbf{resolvent}} \mathbf{f_k}(r)$$

The resolvent $\mathrm{H_k}$ is the *frequency-domain* (so travelling waves) transfer function from nonlinear interaction between scales to velocity field

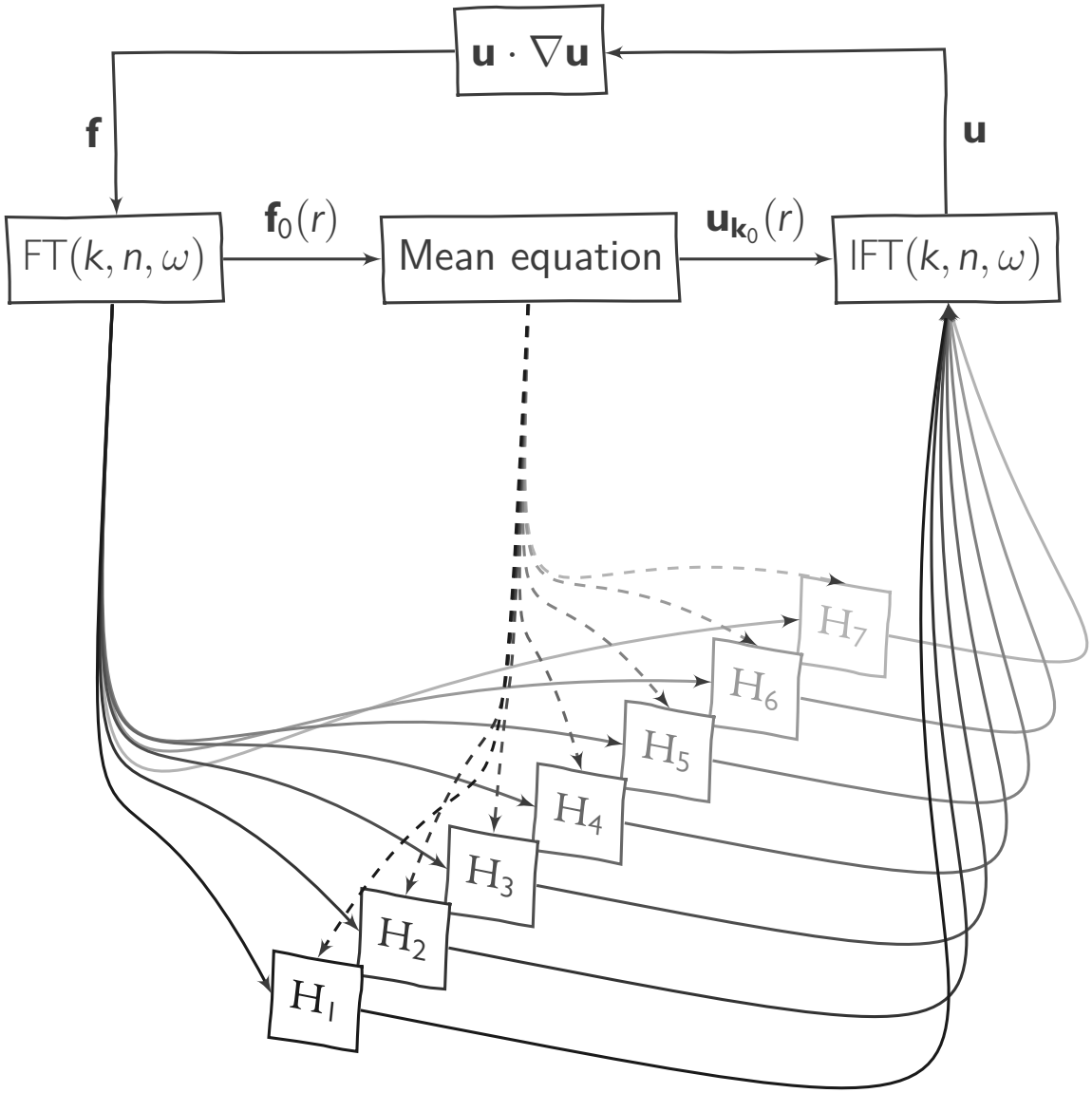
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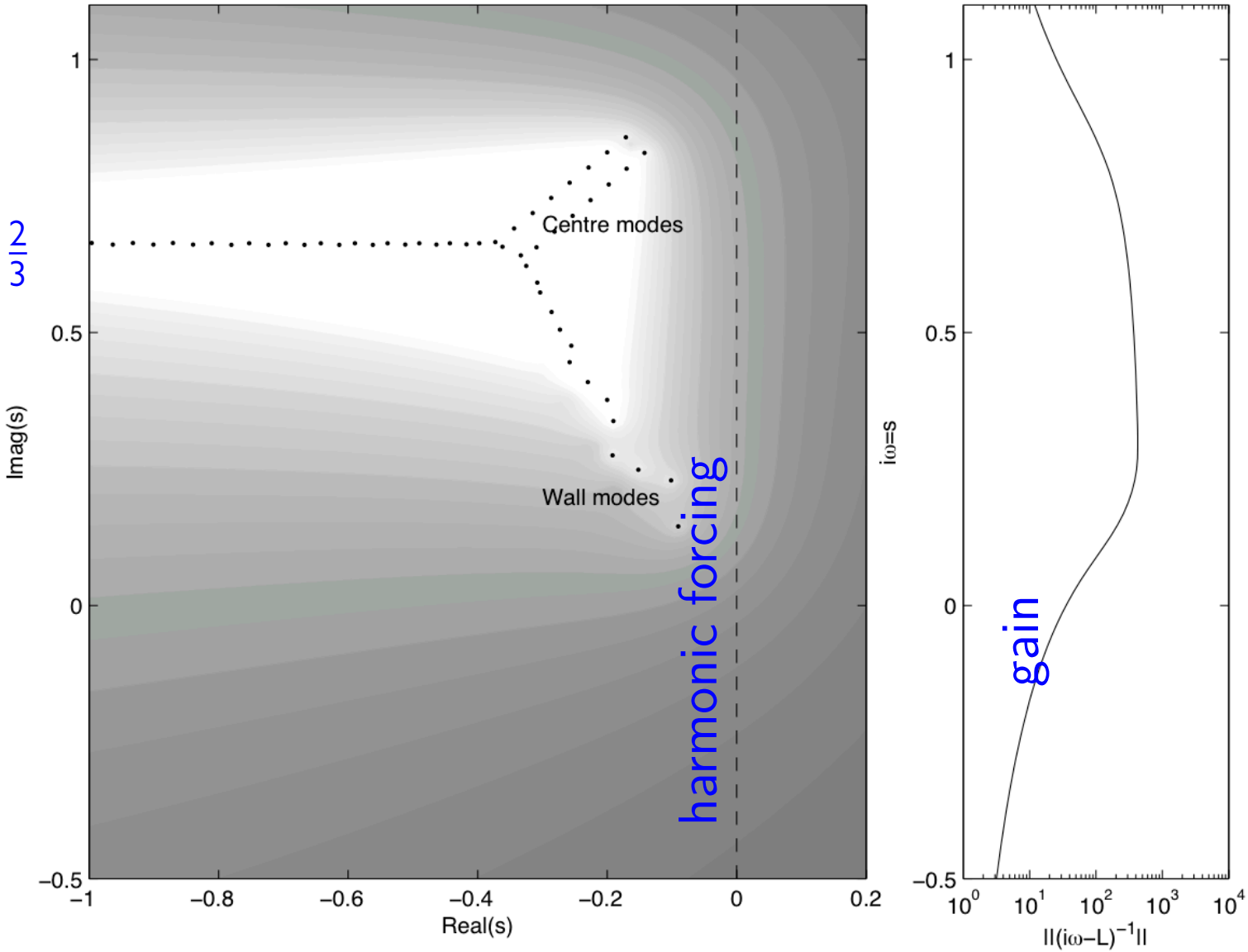
resolvent

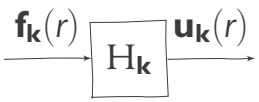
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NSE as a network of resolvents



Local stability of $H_\mathbf{k}$, but large response



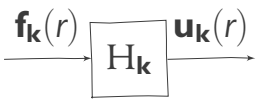


Resolvent

$$\mathrm{H_k} = \begin{bmatrix} ik(\mathbf{u}_0 - c) - Re^{-1}D & -2inr^{-2}Re^{-1} & 0 \\ 2inr^{-2}Re^{-1} & ik(\mathbf{u}_0 - c) - Re^{-1}D & 0 \\ -\partial_r \mathbf{u}_0 & 0 & ik(\mathbf{u}_0 - c) - Re^{-1}(D + r^{-2}) \end{bmatrix}^{-1}$$

$D = \partial_r^2 + r^{-1}\partial_r - r^{-2}(n^2 + 1) - k^2$, states (u_r, u_θ, u_x)

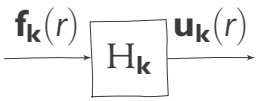
Meseguer & Trefethen JCP 2003



Non-normality

$$\mathrm{H_k} = \begin{bmatrix} ik(\mathbf{u}_0 - c) - Re^{-1}D & -2inr^{-2}Re^{-1} & 0 \\ 2inr^{-2}Re^{-1} & ik(\mathbf{u}_0 - c) - Re^{-1}D & 0 \\ -\partial_r \mathbf{u}_0 & 0 & ik(\mathbf{u}_0 - c) - Re^{-1}(D + r^{-2}) \end{bmatrix}^{-1}$$

- This shear term is main source of non-normality (can lead to large response)
- Important for motions with large radial span
- Only net source of fluctuation energy
- Translational symmetries $\Leftrightarrow$ normal operator



Critical layer response

$$\mathrm{H_k} = \begin{bmatrix} ik(\mathbf{u}_0 - c) - Re^{-1}D & -2inr^{-2}Re^{-1} & 0 \\ 2inr^{-2}Re^{-1} & ik(\mathbf{u}_0 - c) - Re^{-1}D & 0 \\ -\partial_r \mathbf{u}_0 & 0 & ik(\mathbf{u}_0 - c) - Re^{-1}(D + r^{-2}) \end{bmatrix}^{-1}$$

- as $c = \omega/k \to \mathbf{u}_0$ and $Re \to \infty$, $\quad \|\mathrm{H_k}\| \to \infty$
- Energetic response at critical layer becomes very important at high Re.
- decouples from shear mechanism
- response becomes more localised around CL

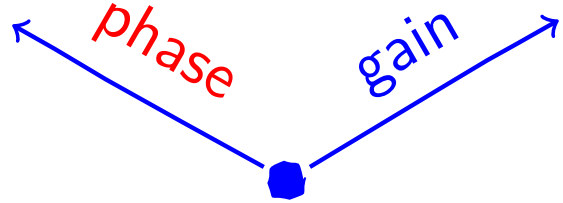
McKeon & Sharma JFM 2010

Approximate $\mathbf{H_k}$ by Gain or phase?

For critical layer, gain is important. For non-normality, phase is important.

The nonlinearity has interesting phase characteristics (is conservative) but is unbounded in gain.

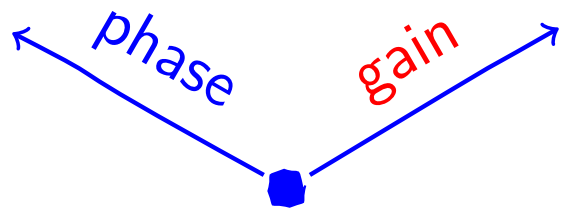
Sharma & al PoF 2011



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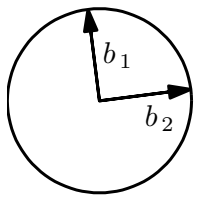
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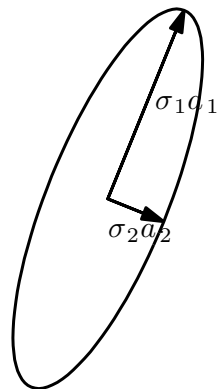
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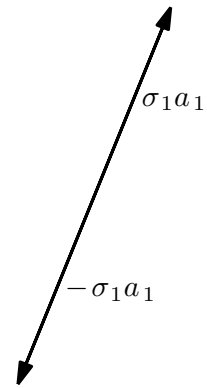


The singular value decomposition

SVD gives L_2 gain-optimal low-dimensional approximation of an operator.





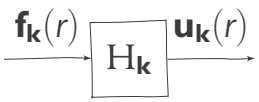


$$M = \begin{bmatrix} 0.5 & 0.9 \\ -0.5 & 2 \end{bmatrix} = ASB^*,$$

$$A = \begin{bmatrix} 0.3761 & 0.9266 \\ 0.9266 & -0.3761 \end{bmatrix}, \quad S = \begin{bmatrix} 2.2089 & 0 \\ 0 & 0.6564 \end{bmatrix}, \quad B = \begin{bmatrix} -0.1246 & 0.9922 \\ 0.9922 & 0.1246 \end{bmatrix}.$$

Optimal rank-1 approximation:

$$\tilde{M} = A \begin{bmatrix} 2.2089 & 0 \\ 0 & 0 \end{bmatrix} B^*$$



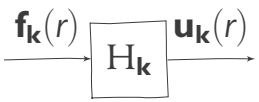
Approximating a single resolvent

SVD approximates an operator by directions of principal gain

$$\mathrm{H}_{\mathbf{k}} = \sum_{m=1}^{\infty} \psi_{m\mathbf{k}}(r)\sigma_{m\mathbf{k}}\phi^*_{m\mathbf{k}}(r)$$

$$\delta_{l,m} = (\phi^*_{l\mathbf{k}}(r), \phi^*_{m\mathbf{k}}(r))_r$$
$$\delta_{l,m} = (\psi^*_{l\mathbf{k}}(r), \psi^*_{m\mathbf{k}}(r))_r$$
$$\sigma_m \geq \sigma_{m+1} \geq 0$$

Each σ_m is a (real) gain,
σ_1 is the maximum gain.



Approximating a single resolvent

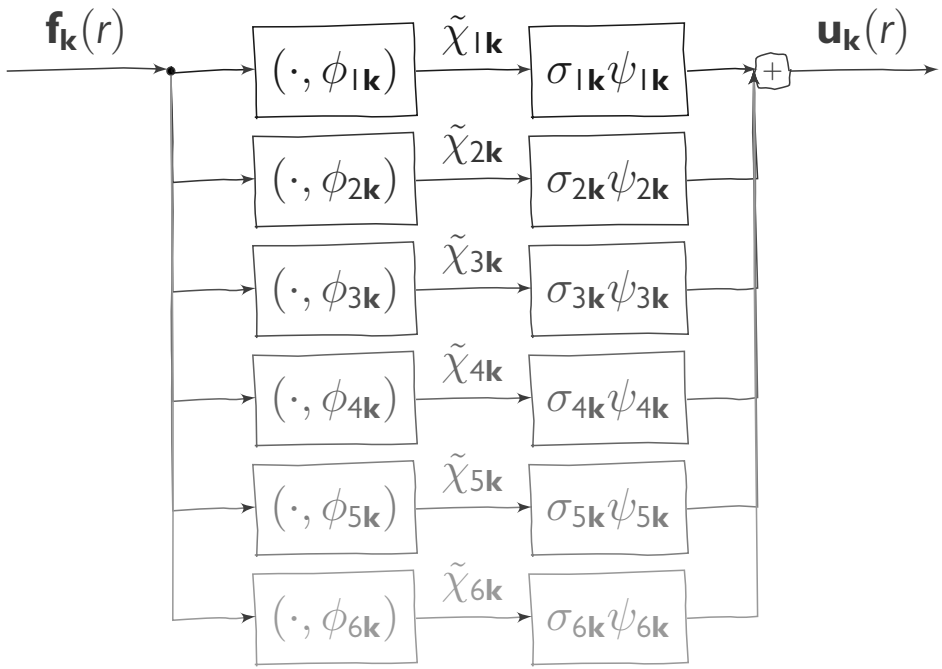
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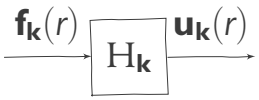
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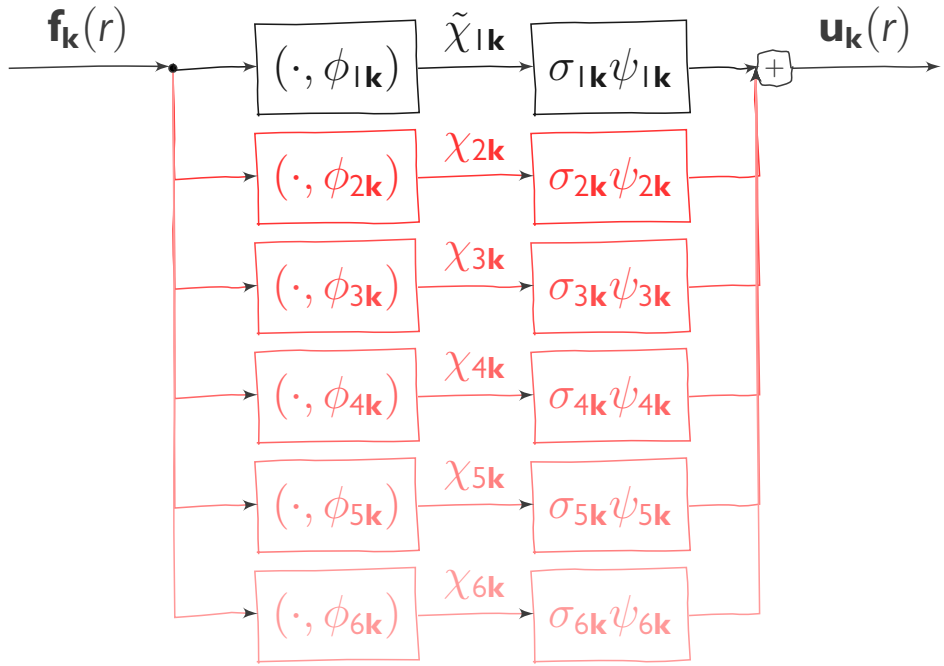


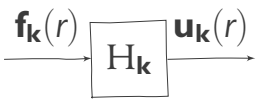


Approximating a single resolvent

$$\mathrm{H}_{\mathbf{k}}\phi_{l\mathbf{k}}(r) = \sigma_{m\mathbf{k}}\psi_{l\mathbf{k}}(r)$$

- maximum gain is achieved by $\mathbf{f_k} = \phi_{l\mathbf{k}}(r)$
- associated response is $\mathbf{u_k} \approx \sigma_{l\mathbf{k}}\psi_{l\mathbf{k}}(r)$
- $\sigma_l \gg \sigma_2$ for many regions of interest





Approximation of $\mathrm{H}_{\mathbf{k}}$ (by gain)

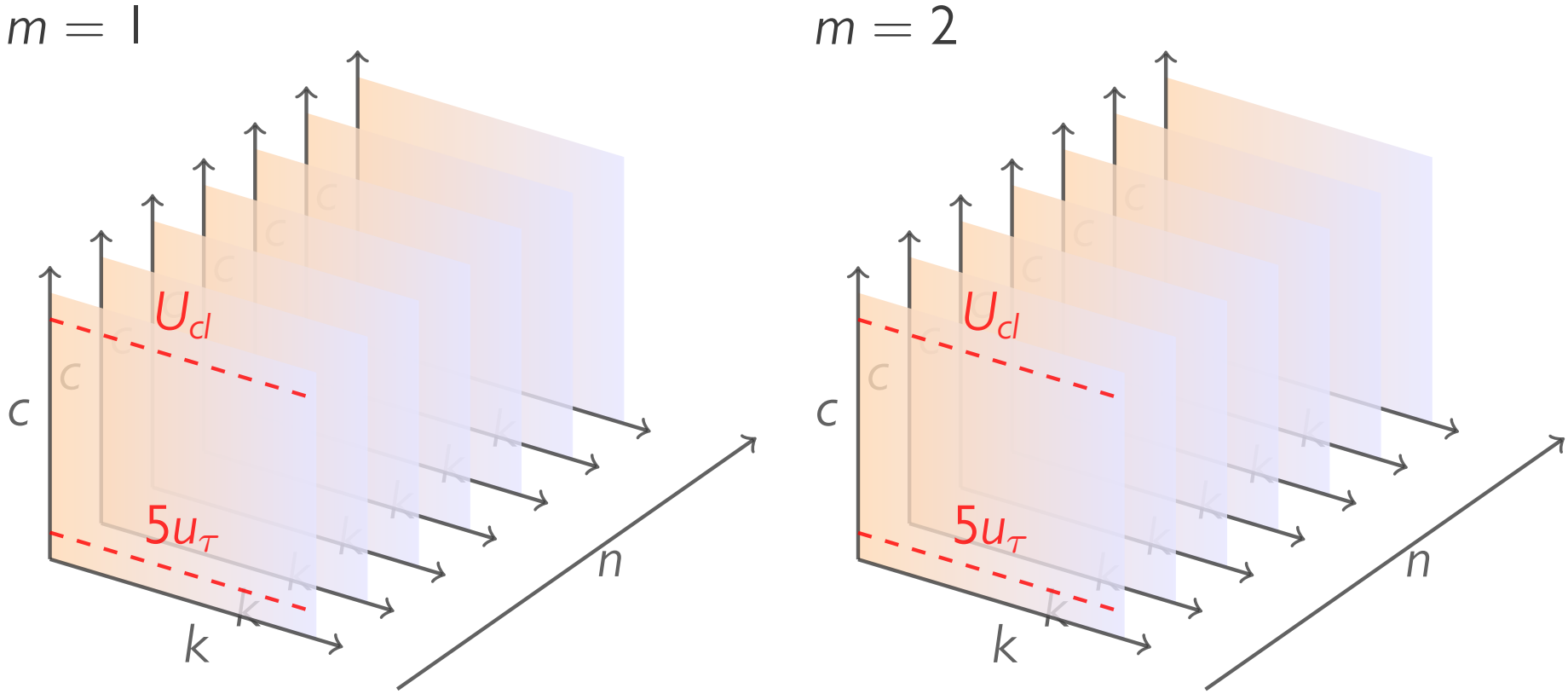
- Optimal rank-1 approximation is the direction of maximum gain
- Since $\sigma_1 \gg \sigma_2$, reasonable to approximate $\mathbf{u_k}$ by $\psi_{1\mathbf{k}}(r)$
- This gives *radial form (or structure)* of velocity field at $\mathbf{k}$
- Reduces NSE solution to a weighted sum of response modes.

- SVD gives alternative basis to Fourier for wall-normal direction
- SVD in x, θ, t recovers Fourier
- Reduces NSE solution to a weighted sum of response modes

$$\mathbf{u}(x, \theta, r, t) \approx \sum_{\mathbf{k}} \chi_{l\mathbf{k}} \psi_{l\mathbf{k}}(r) \mathrm{e}^{i(kx+n\theta-\omega t)}$$

- Great simplification, but remains to find coefficients $\chi_{l\mathbf{k}}$

Turbulence as a hypercube of coefficients



- Truncate outside $5u_\tau < c < U_{cl}$; above high k, n, ω; $m \lesssim 2$
- need for 4D data (DNS / experiment)

Finding coefficients

- Coefficients are fixed by $-\mathbf{u} \cdot \nabla \mathbf{u}$ in a huge quadratic equation

$$\chi_j = \sum_{ab} \sigma_j N_{jab} \chi_a \chi_b$$

- Would be interesting to ponder solutions, but for our purposes two choices:
  1. assume unit forcing for all $\mathbf{k}$
  2. or manually pick combinations of $\chi_{\mathbf{k}}$

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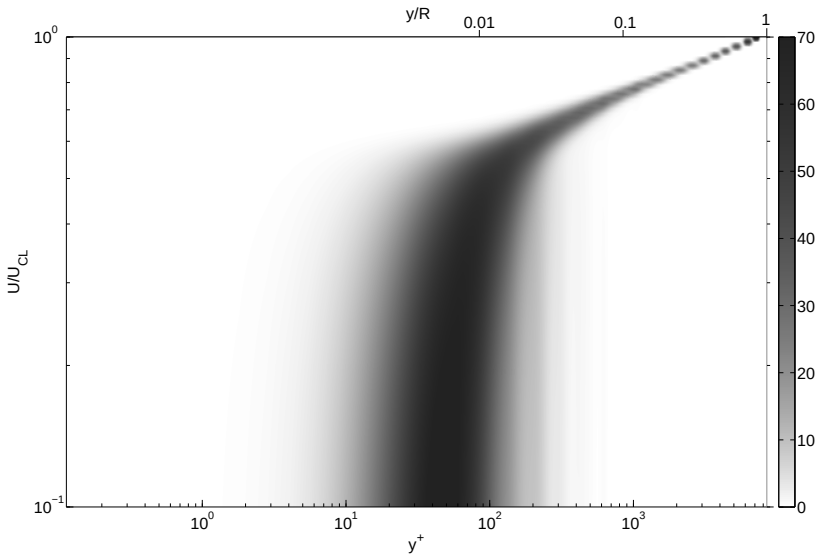
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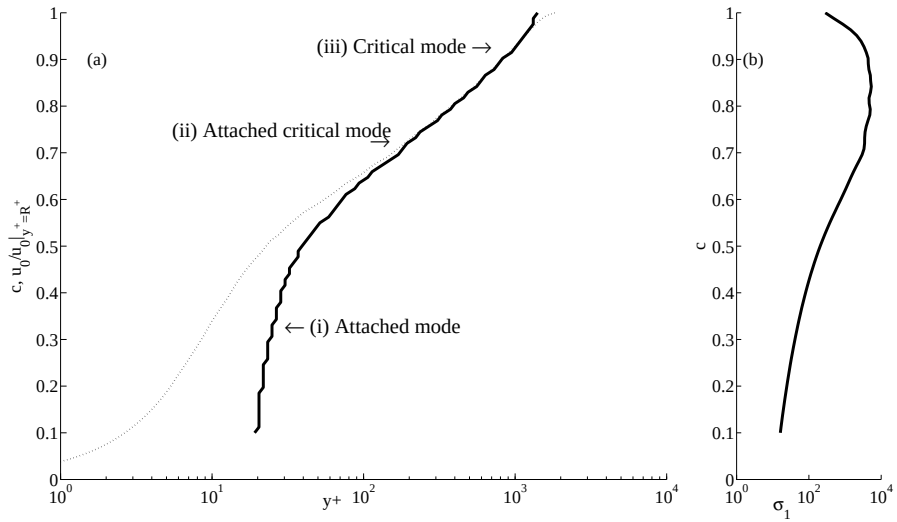
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$$\chi_j = \sum_{ab} \sigma_j N_{jab} \chi_a \chi_b$$

- Would be interesting to ponder solutions, but for our purposes two choices:
  1. assume unit forcing for all $\mathbf{k}$
  2. or manually pick combinations of $\chi_{\mathbf{k}}$ — structure

Response mode shapes, $(k, n) = (1, 10)$

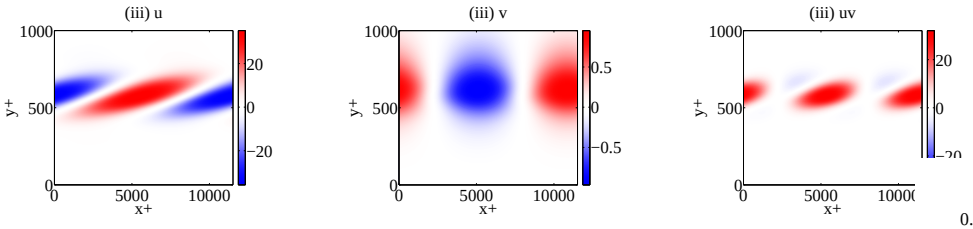




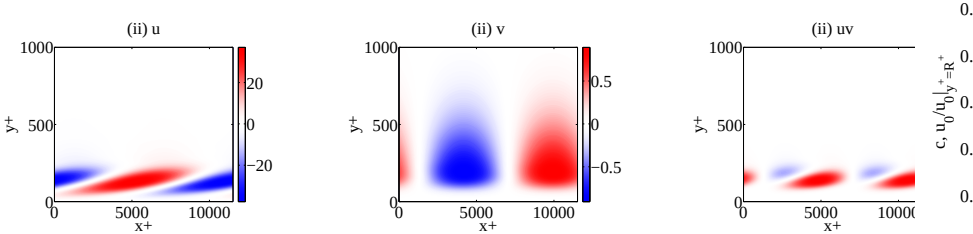
E_{uu}

Response mode shapes, $(k, n) = (1, 10)$

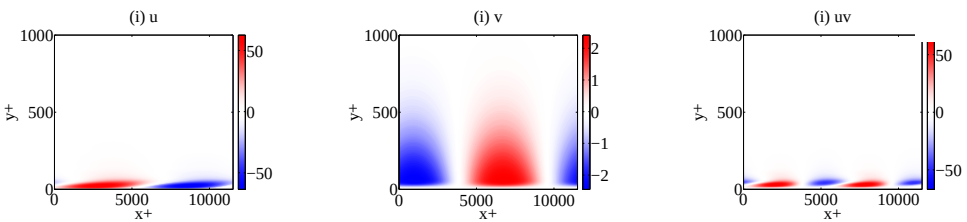
(iii), $c = 0.85$

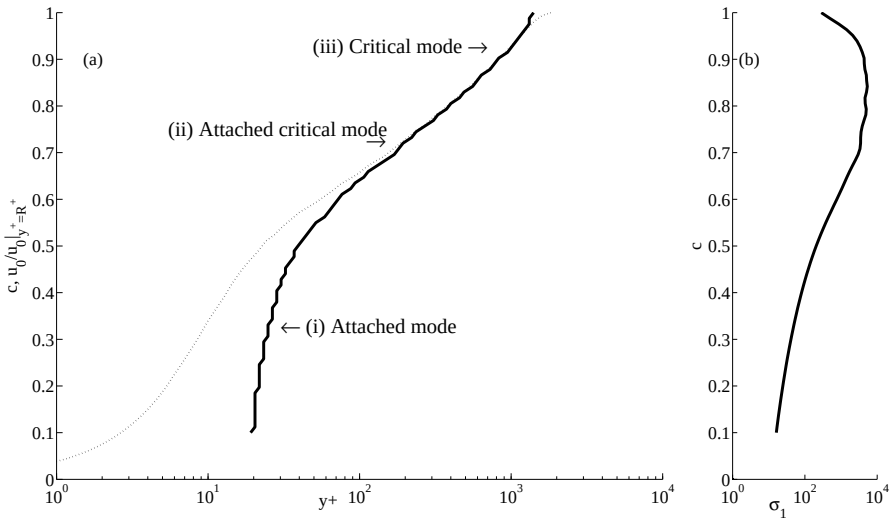


(ii), $c = 0.67$



(i), $c = 0.25$





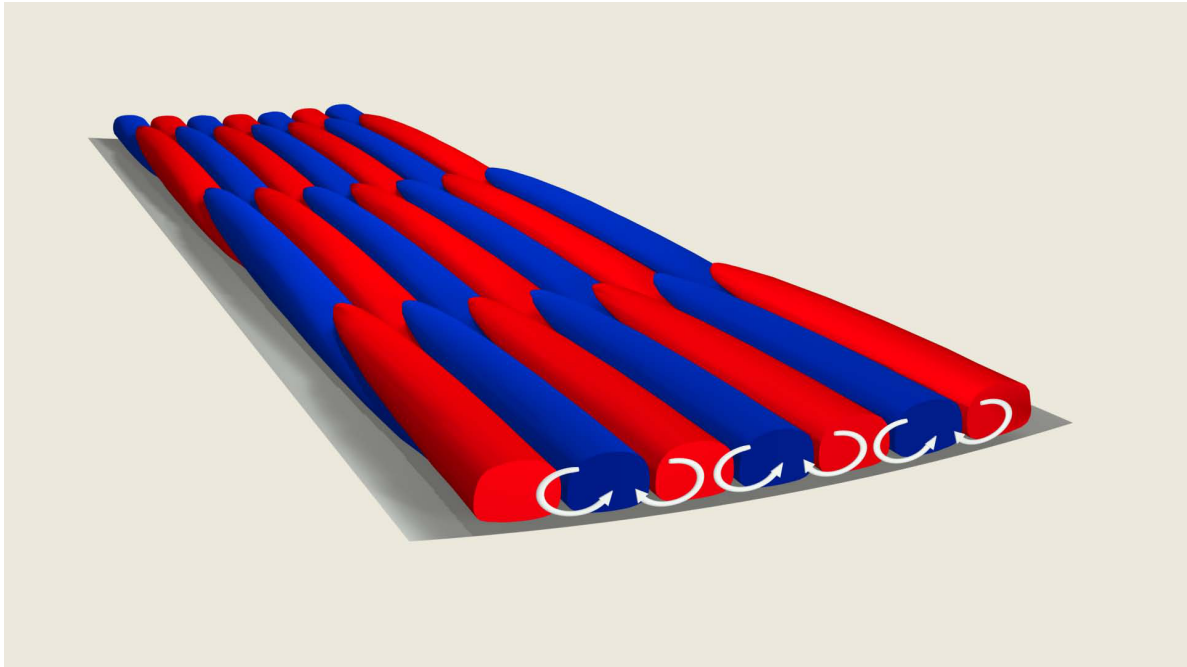
Localisation of response at critical layer

UNIVERSITY OF Southampton

- Forcing & response modes localised in r at critical layer
- Self-similar geometric scaling of dominant modes iff log region
- These modes attached to wall but critical

Example structure

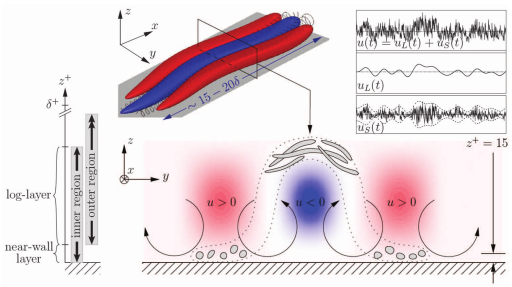
- near-wall cycle
- VLSMs
- hairpin packets...



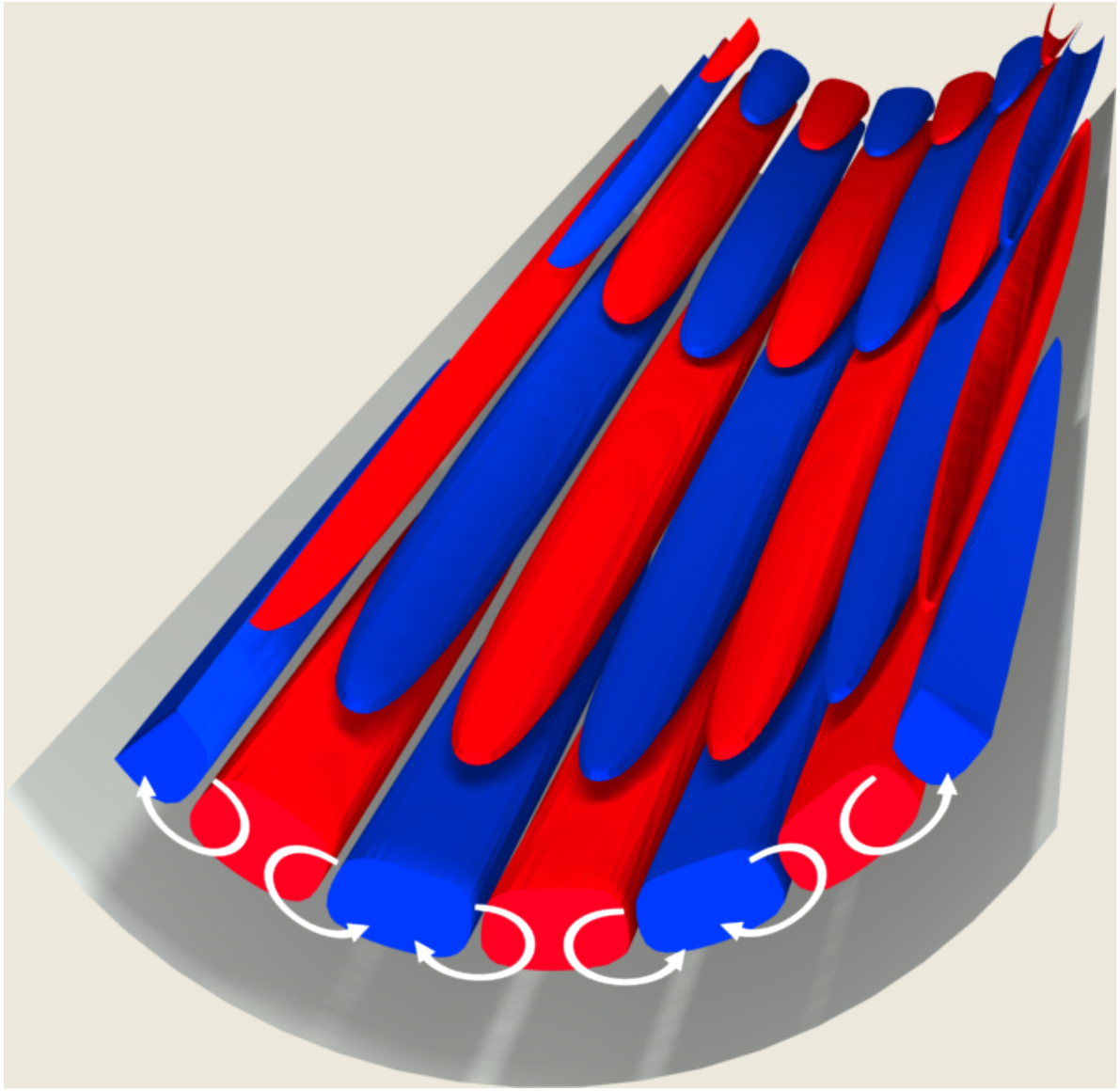
$\psi_1, \ (\lambda_x^+, \lambda_z^+, c^+) = (1000, \pm 100, 10)$

Example structure

- near-wall cycle
- VLSMs
- hairpin packets...



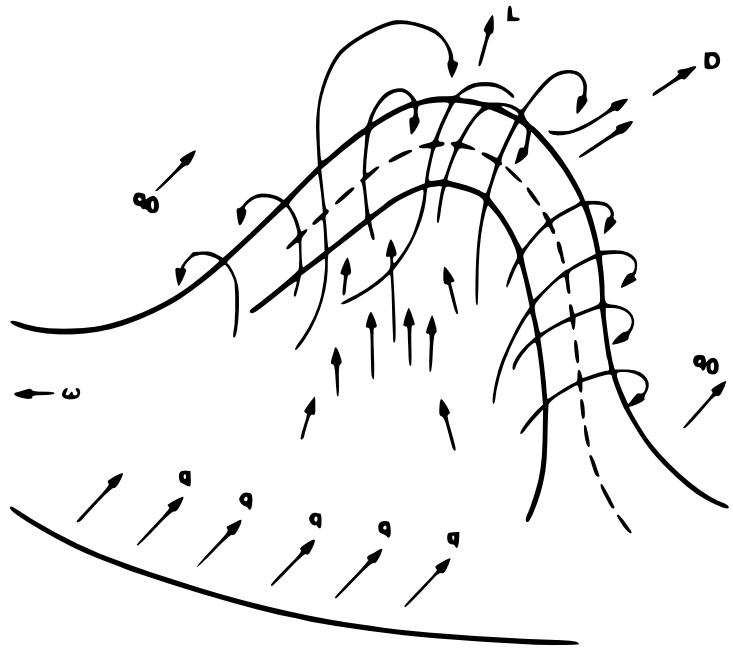
Marusic & al Science 2010



McKeon & Sharma JFM 2010

Example structure

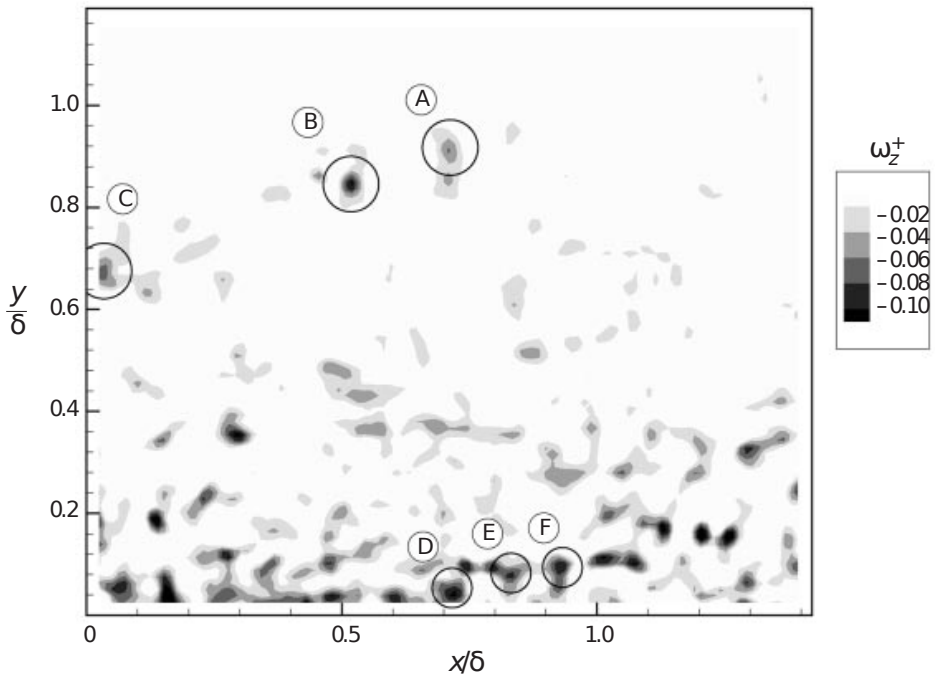
- near-wall cycle
- VLSMs
- hairpin packets...



Theodorsen 1952

Hairpin packets in experiment

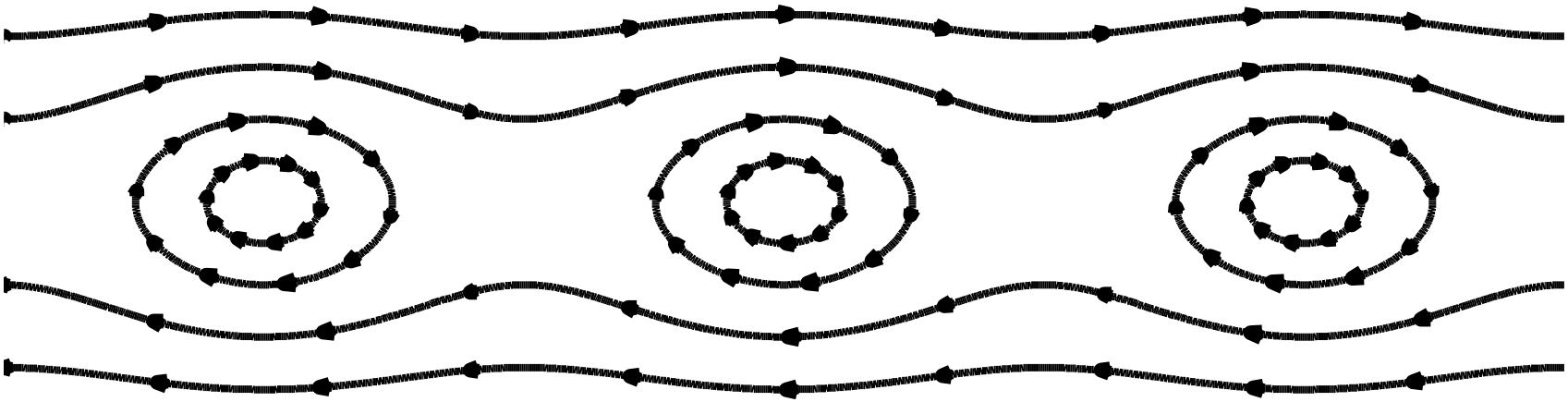
- Adrian & al observed hairpins
- aspect ratio about 1:1
- arranged around long streaks of similar width
- this gives representative (k, n)



Adrian, Meinhart & Tomkins 2000

also Adrain, Christiensen & Liu 2000

Kelvin-Stuart vortices and hairpins

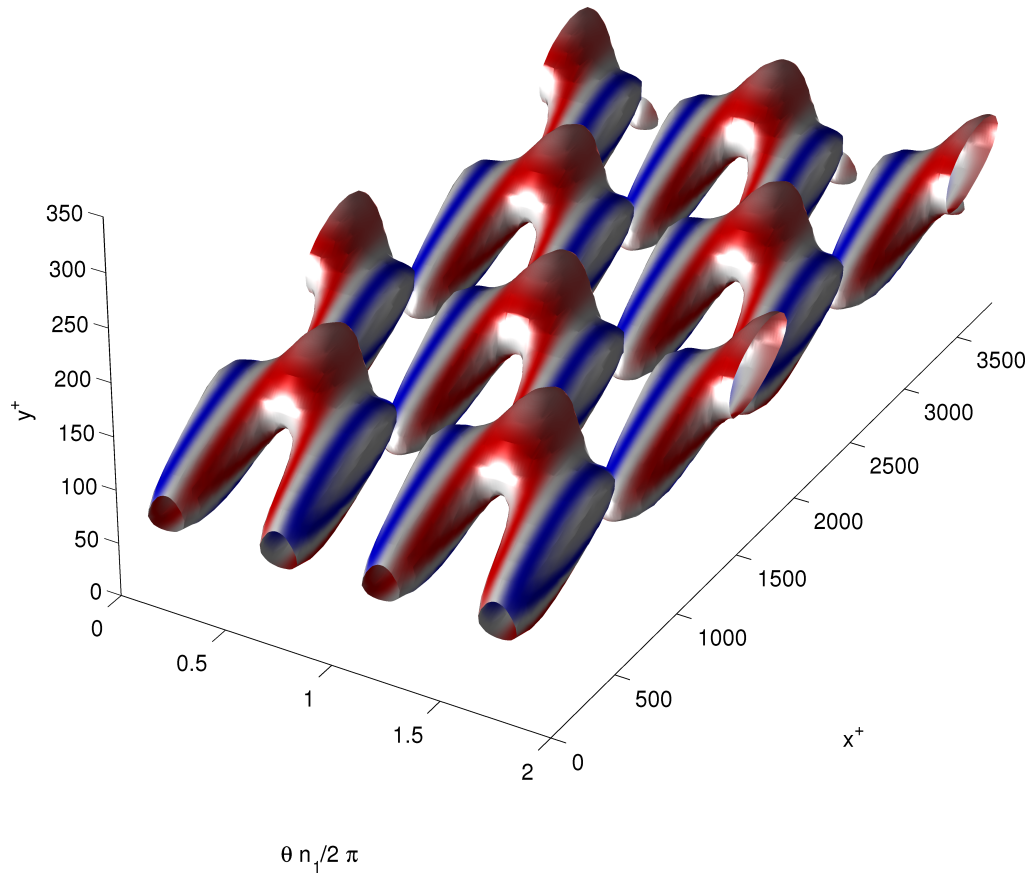


- Hairpins are essentially (the viscous 3D version of) two obliquely intersecting 2D Kelvin-Stuart vortices
- Heads located near the critical layer
- Construct again as pair of left + right going modes

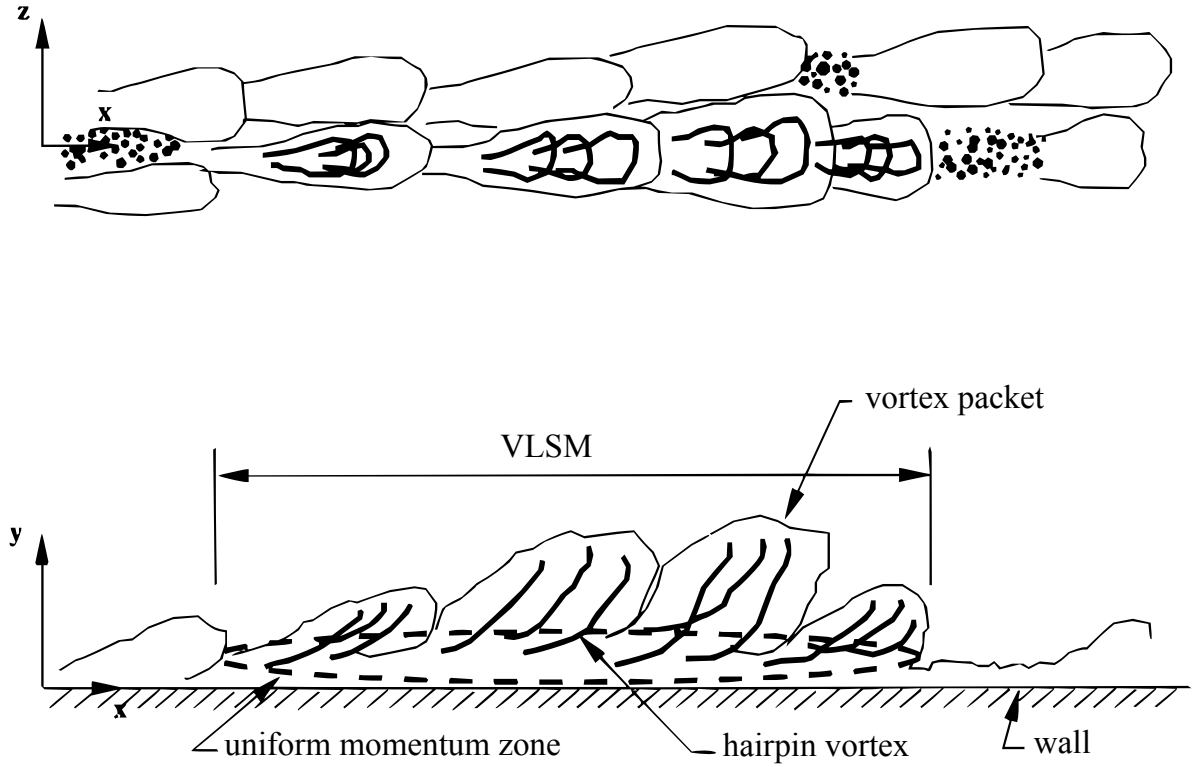
Pairs of modes + mean = 'hairpins'

	k	n	c
mean	0	0	-
'hairpin'	6	± 6	2/3

- mean shear superposes net azimuthal vorticity
- reinforces hairpin heads



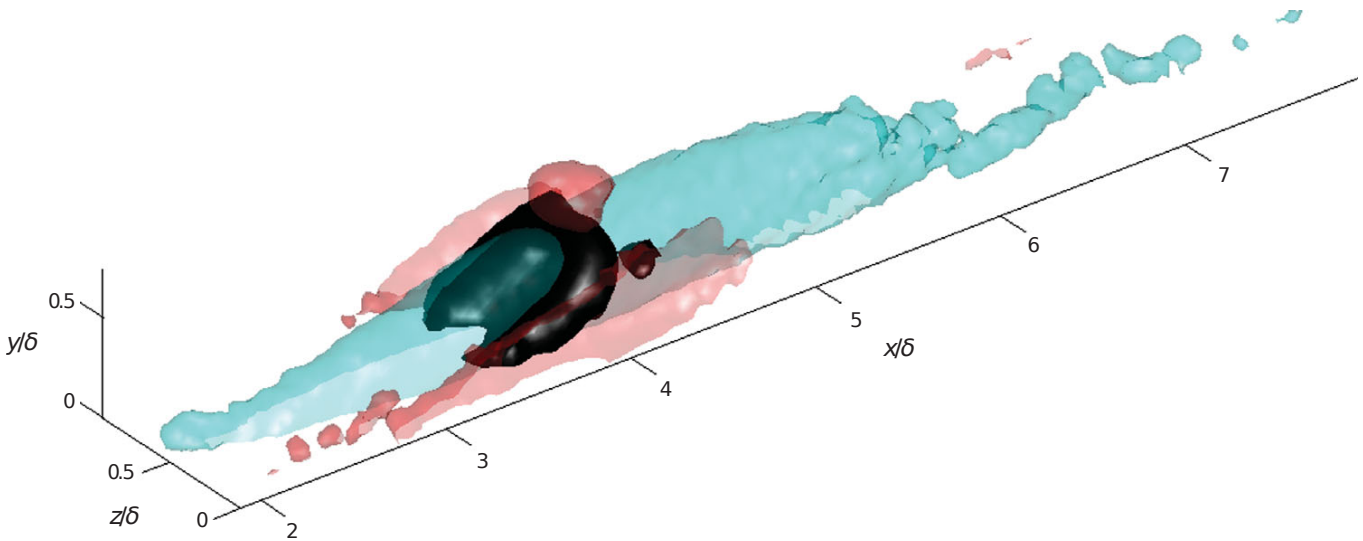
Sharma & McKeon 2013



"Hairpins align coherently in groups to form long packets, and packets align coherently to form very large-scale motions."

Kim & Adrian JFM 1999

- Dennis & Nickels see "clear evidence" of hairpins straddling slower streaks
- assumed to convect at same velocity (Taylor's hypothesis)



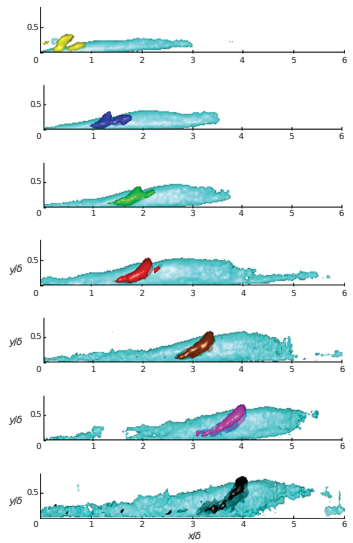
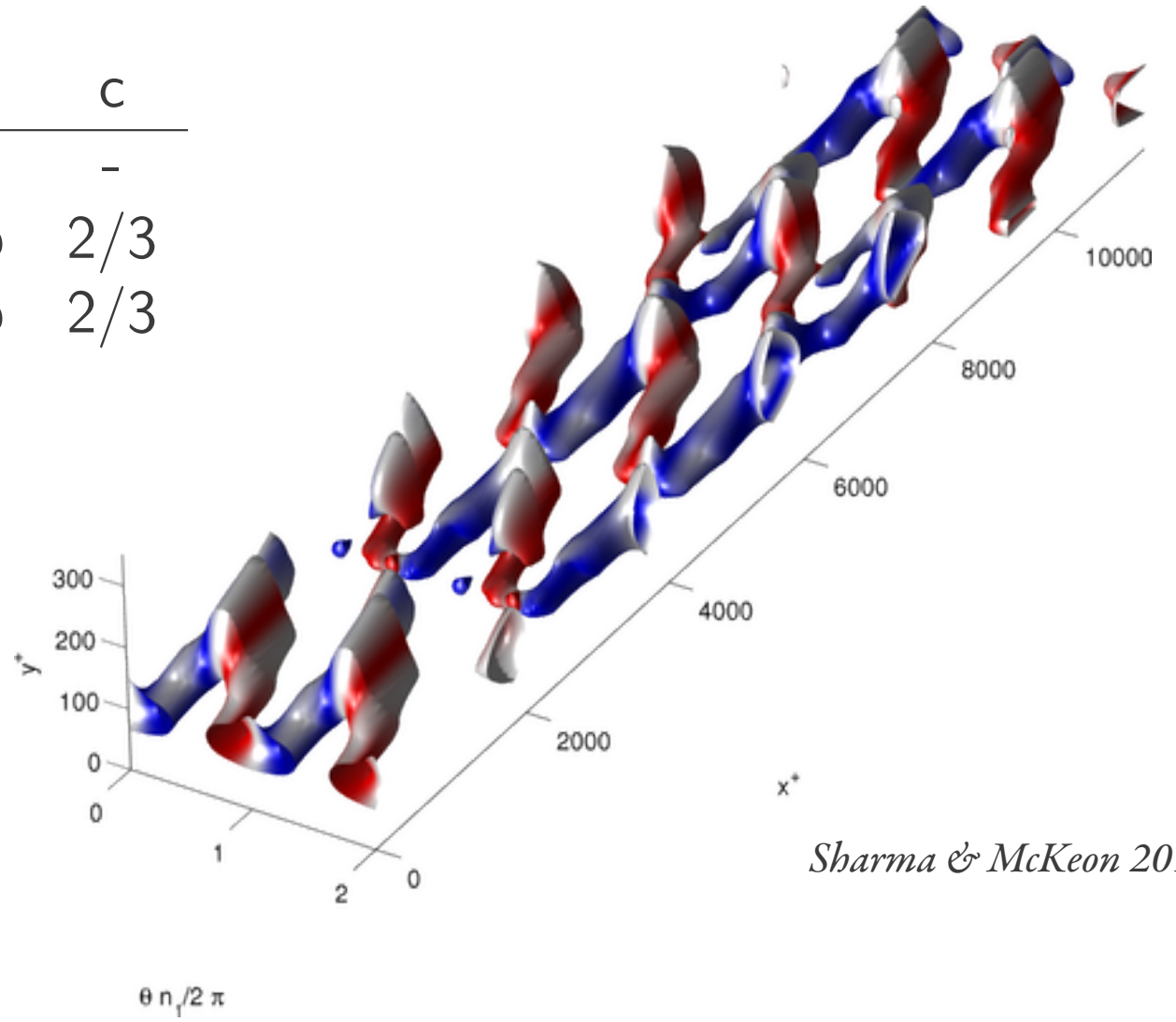


Figure 18. (Colour online) Conditionally averaged field given spanwise swirl. Condition is for $\lambda_{ciz} > 0.18\lambda_{ciz,max}$ at $y/\delta = 0.42$. Black: $\lambda_{ci,iso} = 0.08\lambda_{ci,max}$, blue: $u_{iso} = -0.1u_{max}$, red: $u_{iso} = +0.1u_{max}$.

Dennis & Nickels JFM 2011

‘Hairpins’ + ‘VLSM’ = ‘packet’

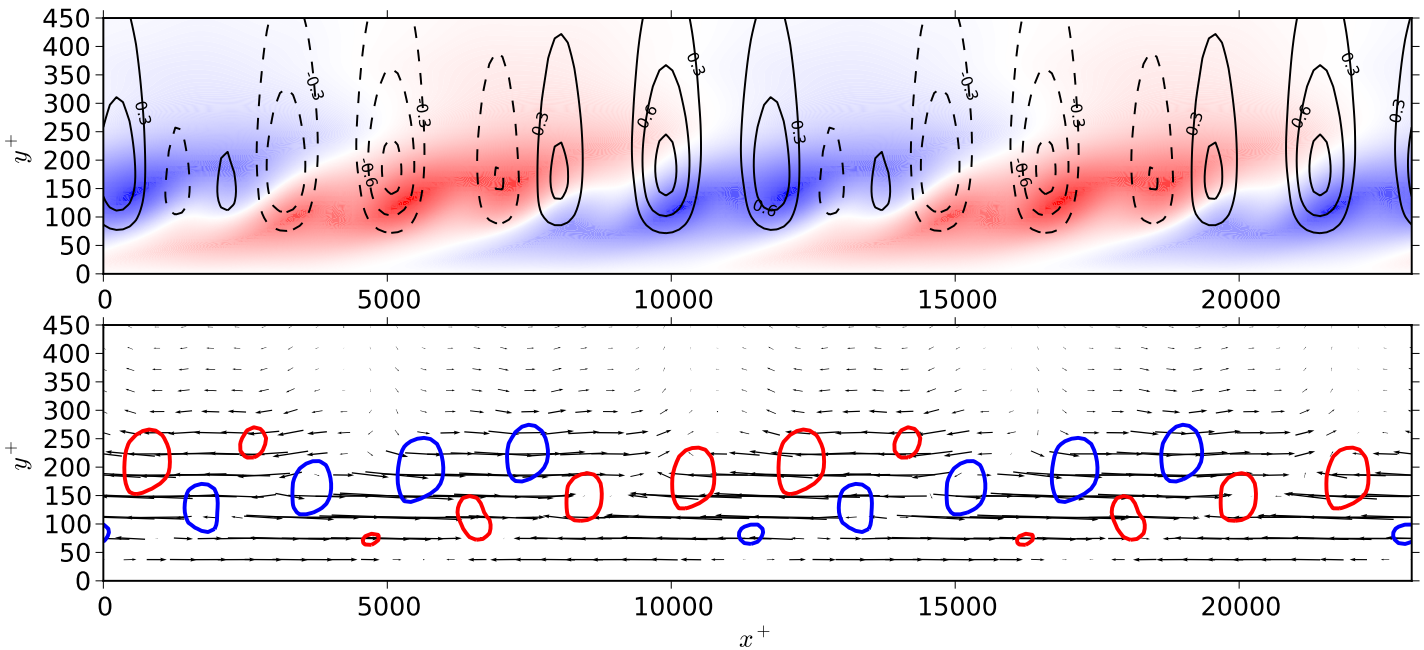
	k	n	c
mean	0	0	-
‘hairpin’	6	± 6	2/3
‘VLSM’	1	± 6	2/3



Sharma & McKeon 2013

Why Taylor's hypothesis?

- we want to match observations
- observations assume Taylor's hypothesis
- required for coherence to be stable over time ('frozen')
- localisation in r implies similar c $(c = 2/3)$



Alignment with VLSMs

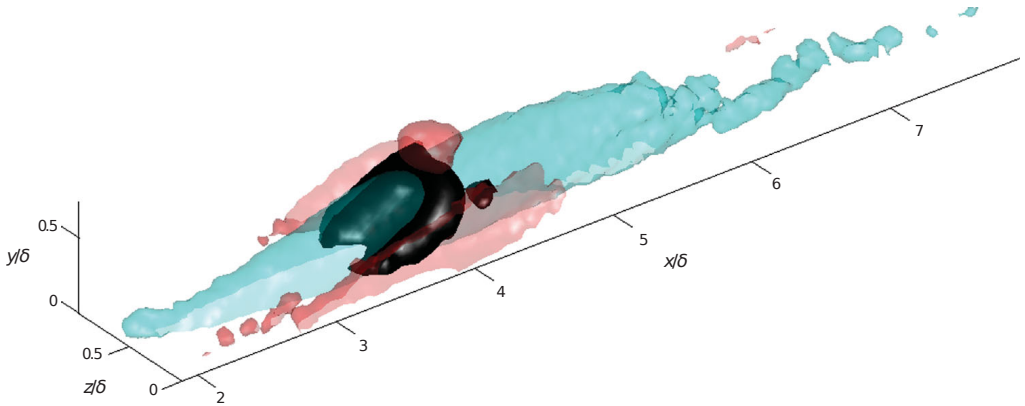
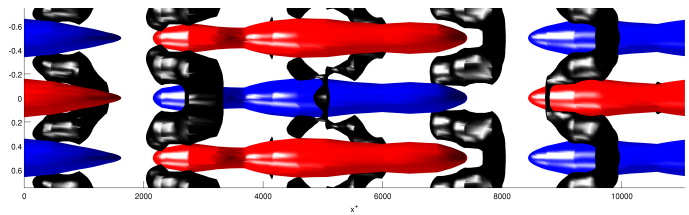
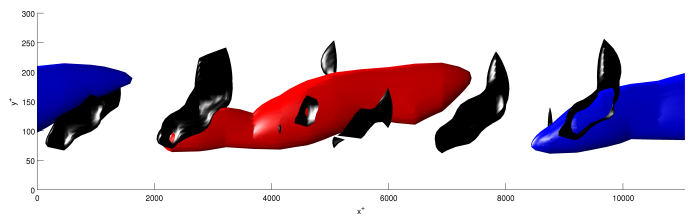
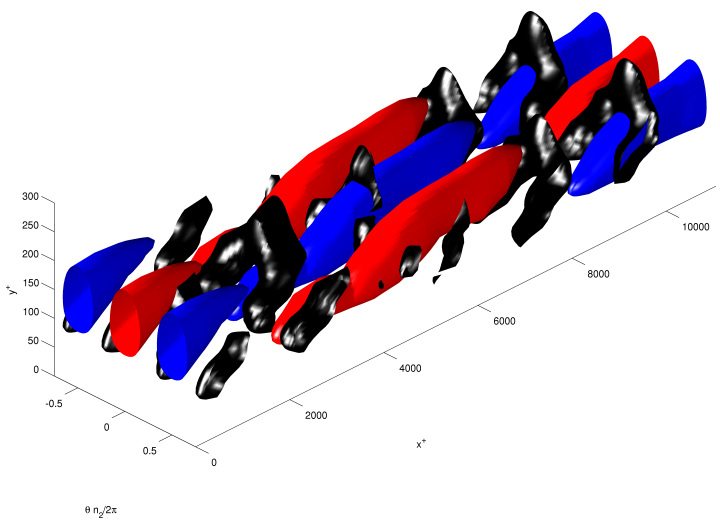


Figure 18. (Colour online) Conditionally averaged field given spanwise swirl. Condition is for $\lambda_{ciz} > 0.18\lambda_{ciz,max}$ at $y/\delta = 0.42$. Black: $\lambda_{ci,iso} = 0.08\lambda_{ci,max}$, blue: $u_{iso} = -0.1u_{max}$, red: $u_{iso} = +0.1u_{max}$.

Dennis & Nickels JFM 2011







Sharma & McKeon 2013

Alignment with VLSMs

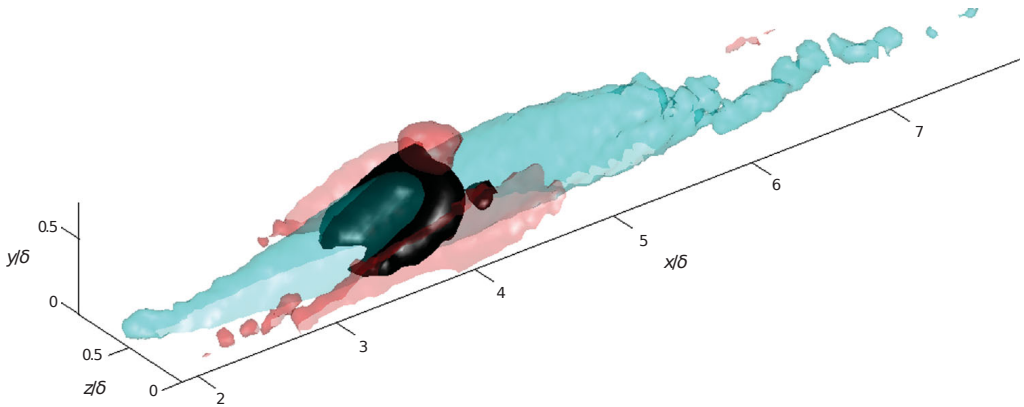
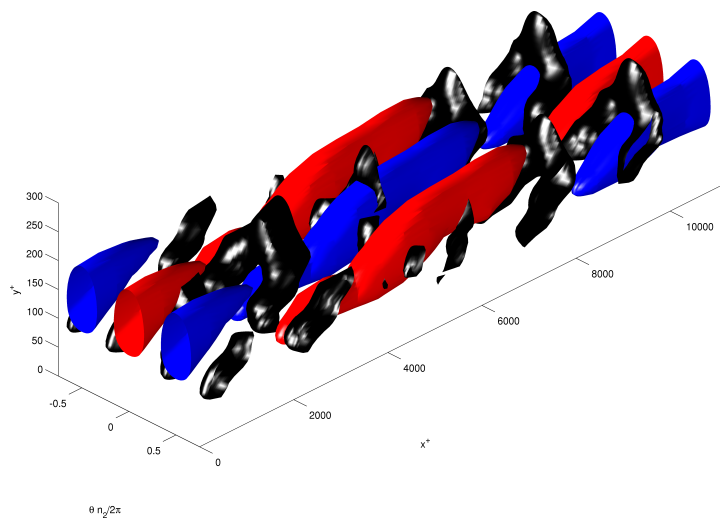


Figure 18. (Colour online) Conditionally averaged field given spanwise swirl. Condition is for $\lambda_{ciz} > 0.18\lambda_{ciz,max}$ at $y/\delta = 0.42$. Black: $\lambda_{ci,iso} = 0.08\lambda_{ci,max}$, blue: $u_{iso} = -0.1u_{max}$, red: $u_{iso} = +0.1u_{max}$.

Dennis & Nickels JFM 2011



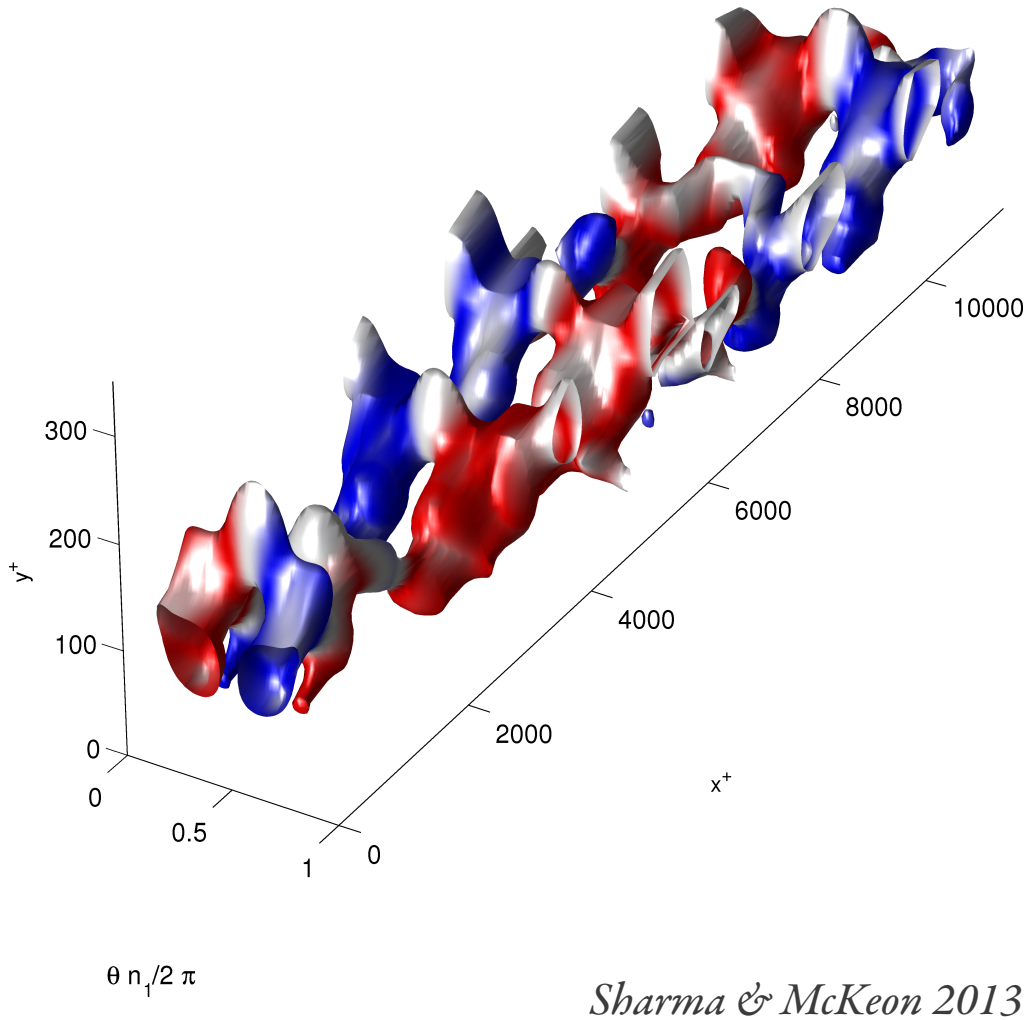
Sharma & McKeon 2013

- VLSMs organise by providing regions of enhanced/reduced shear
- Straddling is artefact of nonlinear swirl diagnostic
- Hairpins appear to organise around regions of low speed

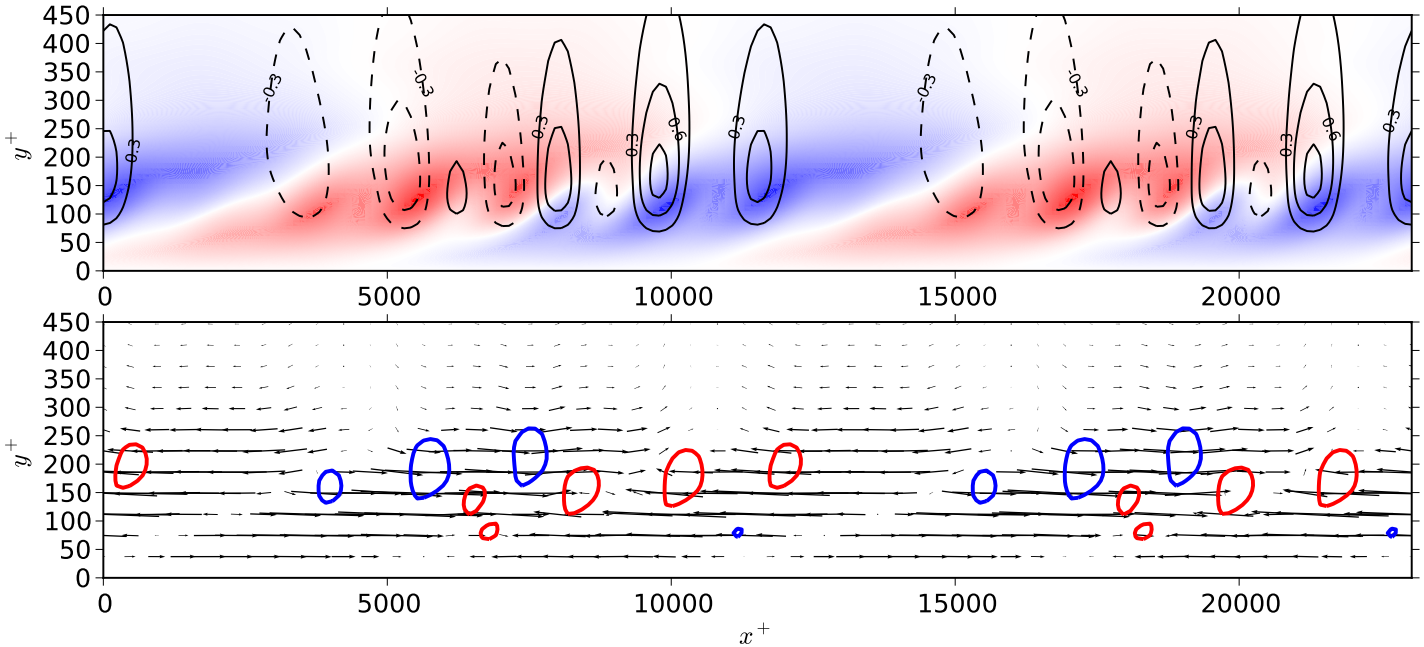
‘Hairpins’ + ‘VLSM’ + beat = ‘modulated packet’

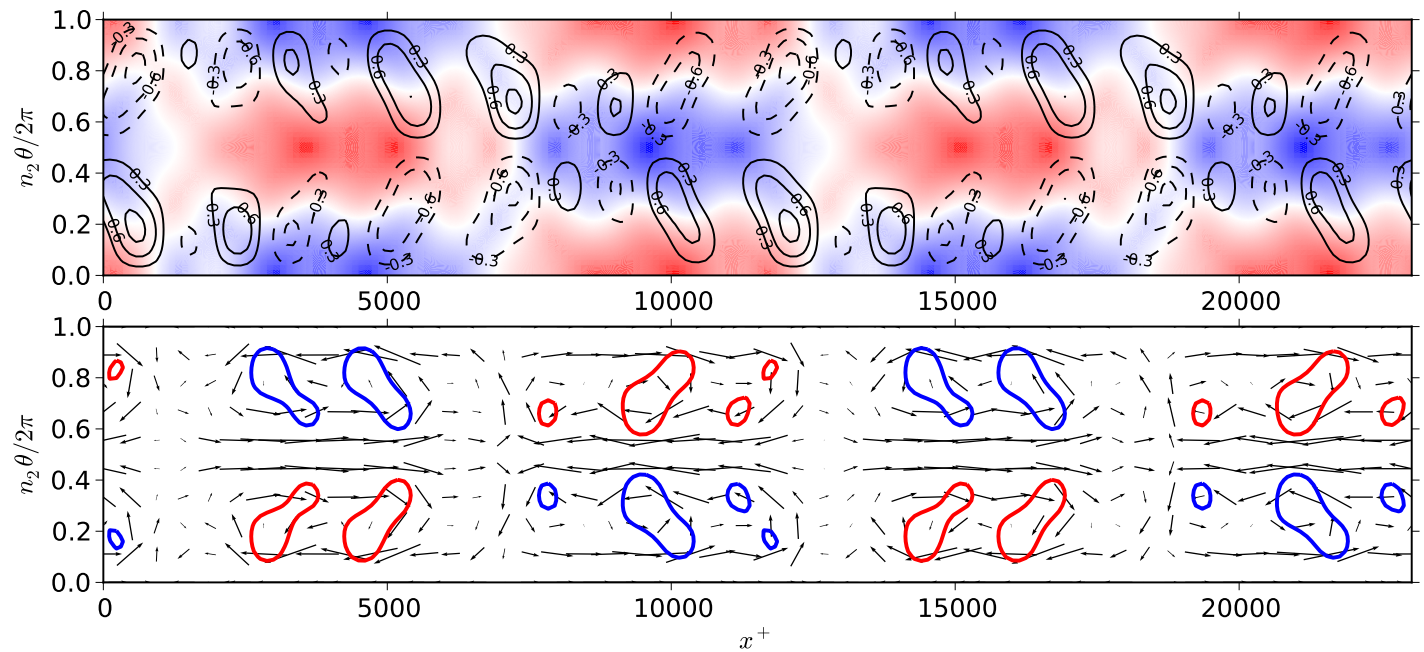
	k	n	c
‘hairpin’	6	± 6	$2/3$
‘VLSM’	1	± 6	$2/3$
beating mode	7	± 12	$2/3$

- VLSM provides local shear to organise hairpins
- A ‘beat’ effect modulates hairpins to form packets

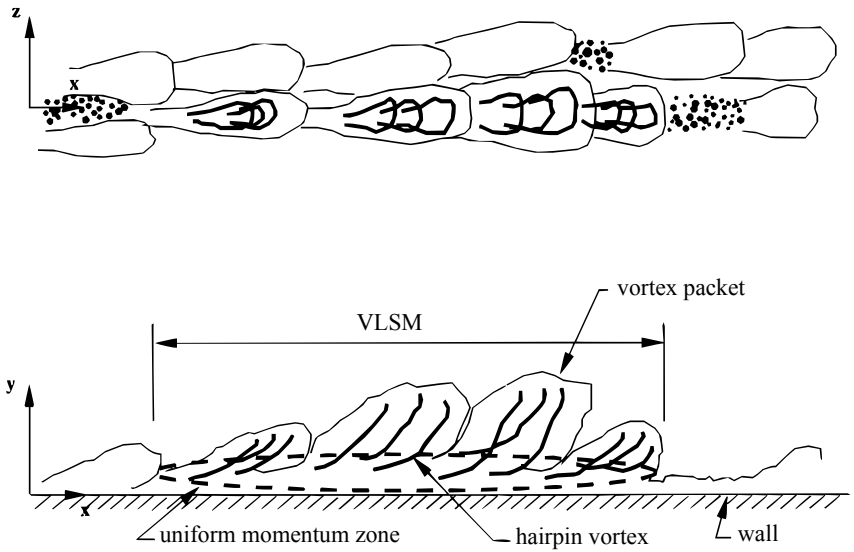


Sharma & McKeon 2013


y^+
x^+
0
5000
10000
15000
20000
0
50
100
150
200
250
300
350
400
450


$n_2\theta/2\pi$
x^+
0
5000
10000
15000
20000
0.0
0.2
0.4
0.6
0.8
1.0

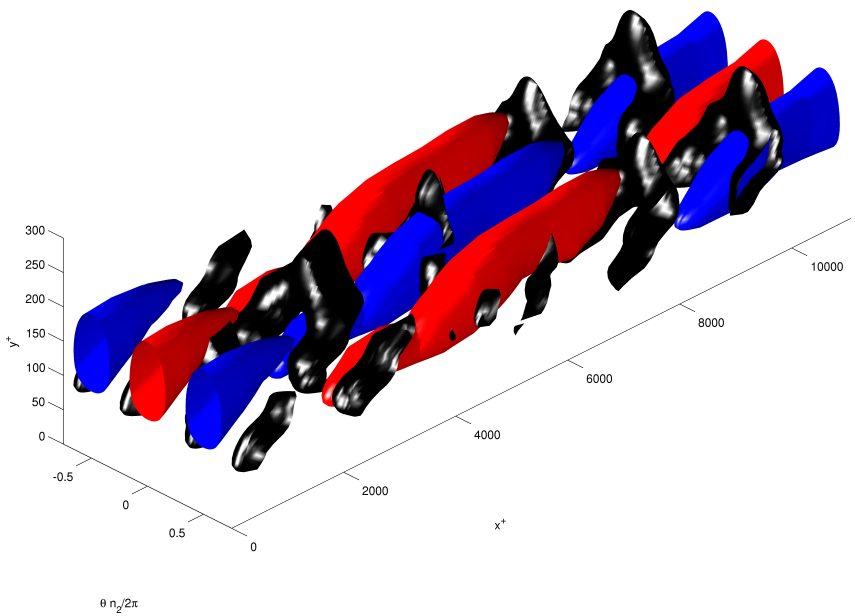
Causality in VLSMs



Kim & Adrian JFM 1999

- Hairpins don't *constitute* or *cause* VLSMs
- but they do appear to align

Causality in VLSMs



- Hairpin heads result from resonance at critical layer
- VLSMs + swirl diagnostic organise hairpins
- Are streaks or hairpins the cause? (recalls streaks/rolls)

Hairpin packet analogy to near-wall cycle

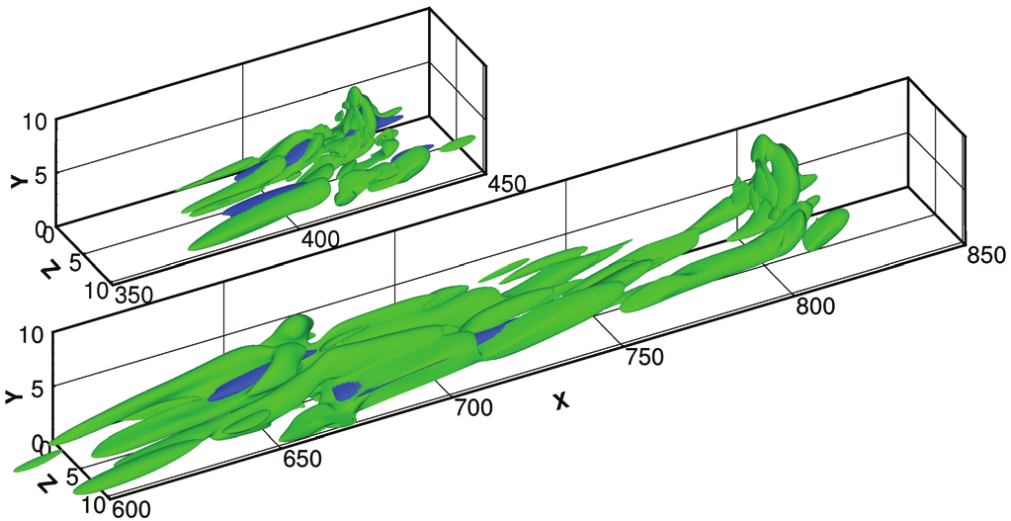


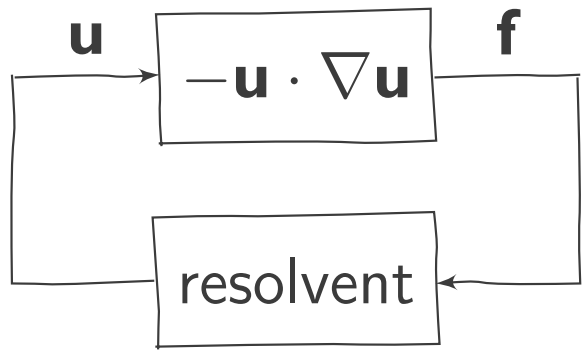
FIG. 2. (Color online) Snapshots of the streamwise component of the perturbation (darker surfaces, blue online, for u ¼ À0:13) and of the Q-criterion (lighter surfaces, green online) at t ¼ 300 and t ¼ 700 (top and bottom, respectively) obtained by the DNS initialized with the nonlinear optimal perturbation with E_0 ¼ 0:004444275.

Cherubini & al PoF 2011

- Recently found 'edge states' look like hairpin packets
- Certain mode combinations can self-sustain via the nonlinearity

Self-exciting processes in the model

The nonlinear term $\mathbf{f} = -\mathbf{u} \cdot \nabla \mathbf{u}$ is not bounded so (by small gain theorem) only *small* disturbances around any solution are guaranteed to decay.



This means there is an amplitude for coefficients for which a mode combination must self-excite.

For a mode combination to self-excite, we just require nonlinear forcing terms not to be orthogonal to the resolvent forcing modes.

The nonlinear term

$$\mathbf{f}_{ab} := -\mathbf{u}_a \cdot \nabla \mathbf{u}_b$$

The full quadratic spectral transfer function for $\mathbf{u} = \mathbf{u}_a + \mathbf{u}_b + \mathbf{u}_c + \ldots$ can be built up from this as the sum

$$\mathbf{f} = \sum_{i=\{a,b,c,\ldots\}} \sum_{j=\{a,b,c,\ldots\}} \mathbf{f}_{ij} = \begin{array}{|ccc|c} \mathbf{f}_{aa} & +\mathbf{f}_{ab} & +\mathbf{f}_{ac} & \cdots \\ +\mathbf{f}_{ba} & +\mathbf{f}_{bb} & +\mathbf{f}_{bc} & \\ +\mathbf{f}_{ca} & +\mathbf{f}_{cb} & +\mathbf{f}_{cc} & \\ \hline \vdots & & & \ddots \end{array}$$

Mode interactions (in x, θ, t)

Wave combinations provide forcing at the sum of wavenumbers

$$\mathbf{u}_a + \mathbf{u}_b \rightarrow \mathbf{f}_{a+b}$$

	k	n	$c = \omega/k$
$\mathbf{k}_l$	6	6	$2/3$
$-\mathbf{k}_l$	-6	-6	$2/3$
$\mathbf{k}_{-l+l}$	0	0	$\omega = 0$

Any real mode interacts with the mean

Mode interactions (in x, θ, t)

Wave combinations provide forcing at the sum of wavenumbers

$$\mathbf{u}_a + \mathbf{u}_b \rightarrow \mathbf{f}_{a+b}$$

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$\mathbf{k}_{-l+l}$	0	0	$\omega = 0$

Any real mode interacts with the mean

	k	n	c
$\mathbf{k}_l$	6	6	$2/3$
$\mathbf{k}_2$	1	6	$2/3$
$\mathbf{k}_{l+2}$	7	12	$2/3$

The idealised hairpin packet can self-interact
Just requires that $(\mathbf{f}_a, \phi_a)_r \neq 0$

Mode interaction (in r)

In wall-normal direction, forcing mode $\phi_{\mathbf{k}}(r)$ is not typically orthogonal to $\mathbf{f}_{\mathbf{k}}(r)$

Modes at triadically consistent $\mathbf{k}$ can self-support with large enough amplitude.

Sharma & McKeon AIAA 2013

Mode interaction (wall-normal cascade)

In wall-normal direction, $(\psi_l(r), \phi_m(r))_r \neq \delta_{lm}$...

Production due to a mode combination is (in matrix form)

$$\frac{dE}{dt} = (\mathbf{f}, \mathbf{u}) = \chi^* \Phi^* \Psi \Sigma \chi \approx \sum_m |\chi_m|^2 (\phi_m, \psi_m) \sigma_{1,m}$$

- dominated by leading mode (factor of σ_1)
- response and forcing modes not orthogonal $\Rightarrow$
- transfer of momentum across r and between modes

Conclusions

Non-orthogonality of resolvent modes:

- Conceptual equivalent of triadic interaction in wall-normal direction
- Overlap of response mode shapes responsible for interaction
- describes transfer of momentum in wall-normal direction

Conclusions

Nonlinearity:

- produces the coherence between modes
- fixes the relative amplitude and phase
- permits self-sustaining mode combinations (amplitude dependent)

Conclusions

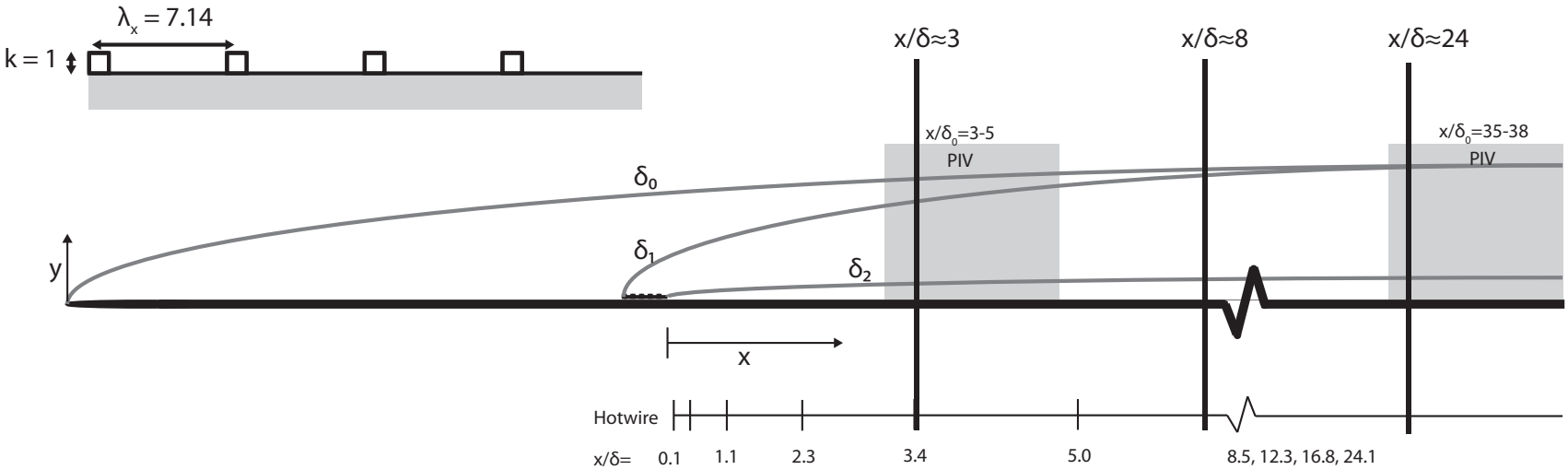
Structure:

- very cheap to calculate
- economic descriptions of flowfields
- low-order models capture structure

Section 6

Related work

Exciting a large-scale motion

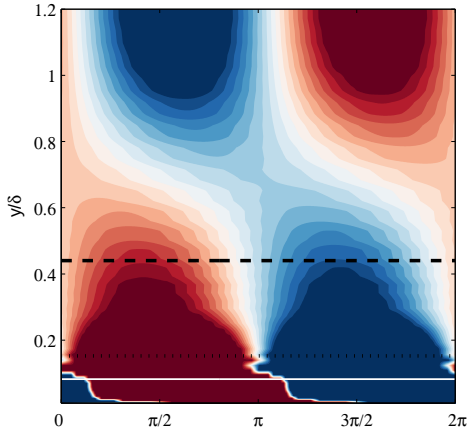


Jacobi & McKeon JFM 2011

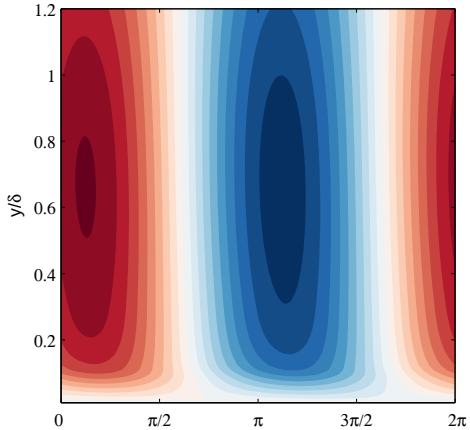
McKeon & al PoF 2013

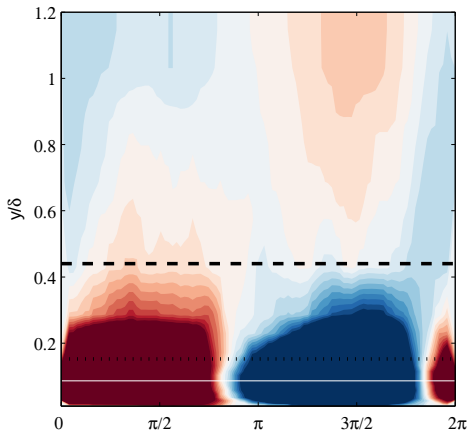
Exciting a large-scale motion

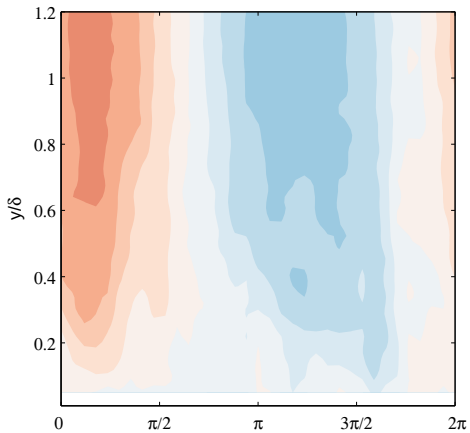
u



v



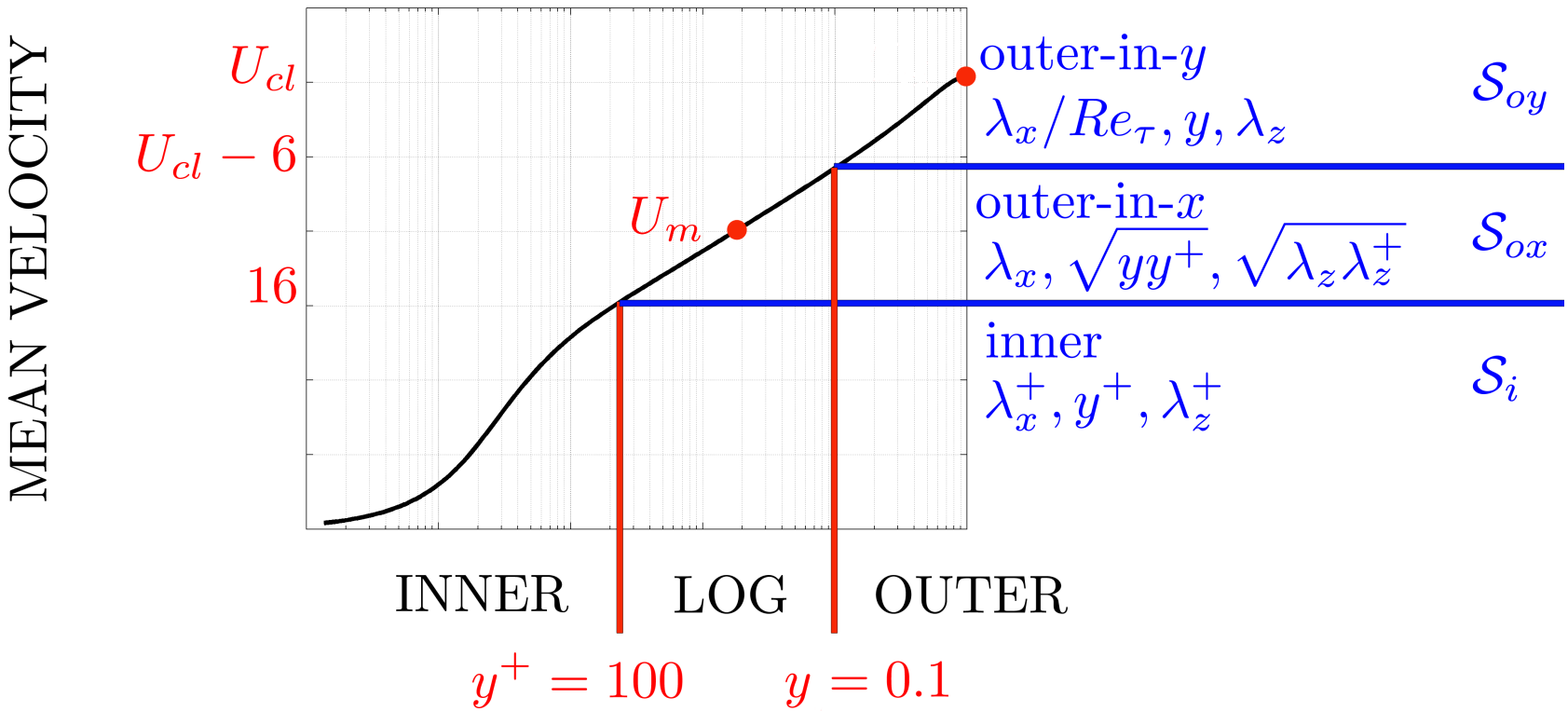




$3\delta_0$ downstream, predicted and measured

McKeon & al PoF 2013

Scaling of modes induced by mean & resolvent

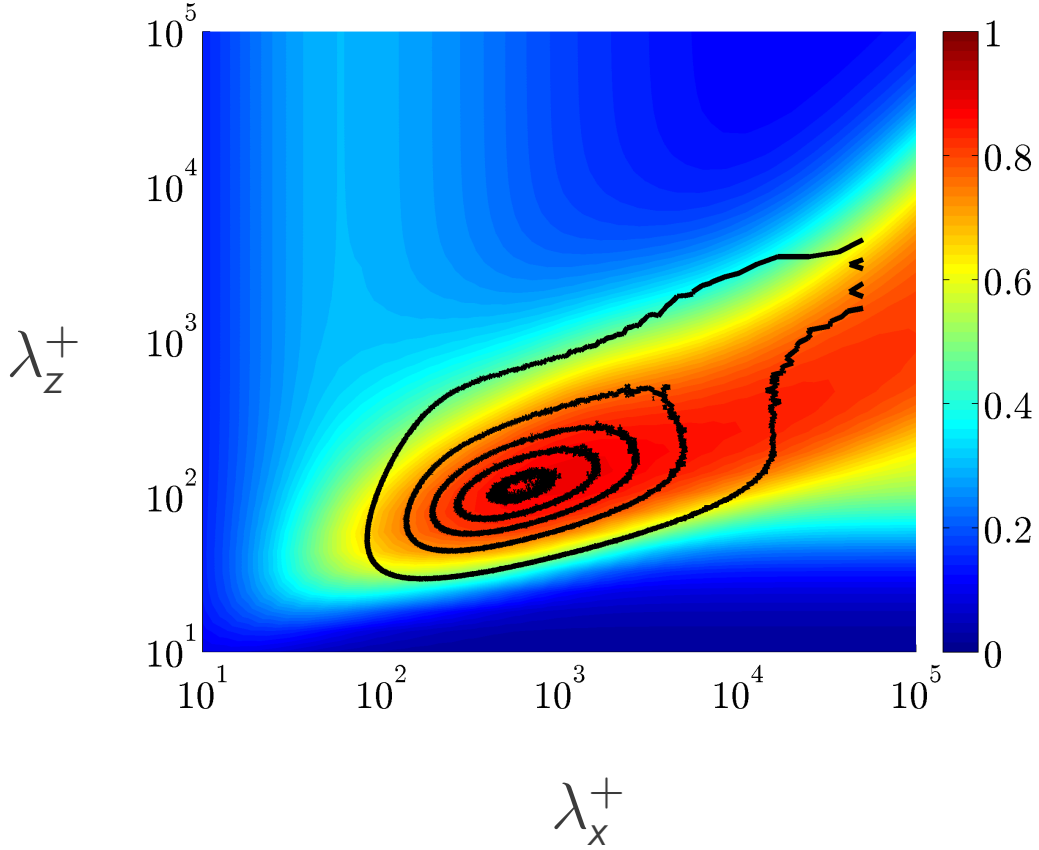


Resolvent analysis reveals Re scaling of response modes and self-similarity in log region.

Moarref & al JFM 2013

Spectra direct from resolvent gains

E_{uu} for unit white noise forcing on first two modes only at $y^+ = 15$ vs DNS



Moarref & al JFM 2013

Hoyas & Jimenez PoF 2006